Revenue Comparisons of Auctions with Ambiguity Averse Sellers

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Abstract

We study the revenue comparison problem of auctions when the seller has a maxmin expected utility preference. We suppose the seller's set of priors satisfies a symmetry property—named rearrangement invariance—around some reference belief, interpreted as an approximation of the true probability law or the focal point distribution. We develop a methodology for comparing the revenue performances of auctions: the seller prefers auction X to auction Y if their transfer functions satisfy a weak form of the single-crossing condition. Intuitively, this condition means that a bidder's payment is more negatively associated with the competitor's type in X than in Y. Applying this methodology, we show that when the reference belief is IID and bidders are ambiguity neutral, (i) the first-price auction outperforms the second-price and all-pay auctions; and, (ii) the second-price and all-pay auctions outperform the war of attrition. Our methodology yields opposite results to the Linkage Principle.

Keywords: Auctions; Ambiguity; Revenue comparison.

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1. Introduction

Since the establishment of the Revenue Equivalence Principle (Myerson, 1981), an important problem of auction theory is to compare the revenue performances of different auctions in setups relaxing Myerson's (1981) standard assumptions. The Linkage Principle (Milgrom and Weber, 1982; Krishna and Morgan, 1997), one of the fundamental results in this direction, states that in the affiliated interdependent values setup, auctions with stronger positive linkages between a bidder's payment and her own signal yield higher expected revenues. Succeeding works study the effects of the bidders' risk aversion (Maskin and Riley, 1984), the seller's risk aversion (Waehrer et al., 1998), the bidders' financial constraints (Che and Gale, 1998), and asymmetric valuation distributions (Maskin and Riley, 2000).

This paper studies the revenue comparison problem when the seller's preference exhibits ambiguity aversion (Ellsberg, 1961). One of our main contributions, Theorem 2, provides a methodology to compare the revenue performances of different auctions. Intuitively, it states that auctions in which a bidder's payment is more negatively associated with the competitor's type yield higher worst-case revenues. Applying this methodology, we compare the revenue performances of four commonly studied auctions: the first-price, second-price, all-pay auctions and the war of attrition.

Following the maxmin expected model (MMEU) of Gilboa and Schmeidler (1989), the seller holds a set of priors around some *reference belief*, and evaluates an auction by the *worst-case revenue*, the minimum expected revenue over the set of priors. The reference belief can be interpreted as an approximating model of the true distribution (Hansen and Sargent, 2001, 2008) or the focal point distribution (Bose et al., 2006; Bose and Daripa, 2009). The set of priors satisfies a symmetry property named *rearrangement invariance* (Definition 1 and Assumptions 1A-1B). This requires, e.g., in the special case of discrete state space and uniform reference belief, that the set of priors remains unchanged under permutations of states (Figure 1). Our assumption incorporates a wide range of examples—most importantly, the relative entropy neighborhood in the robustness literature (Example 1 (a)). To present our results clearly, we primarily focus on the case of ambiguity neutral bidders; however, most of our results extend to the case of the ambiguity averse bidders (Section 6.2).

To develop our main methodology, we first show that in evaluating the worst-case revenue, we can restrict attention to beliefs likelihood ratio dominated by the reference belief—i.e., beliefs that overweight low types and underweight high types relative to the reference belief (Theorem 1). We prove Theorem 1 using two variants of the classical rearrangement inequality in mathematics (Hardy et al., 1959).

Building on Theorem 1, Theorem 2 states that the seller prefers auction X to auction Y if the following two conditions hold. First, each type of bidder's payment is greater (or smaller, resp.) in X than in Y against a competitor with low types (or high types, resp.) (*Weak Single-Crossing Condition, WSCC*; Figure 3). This condition is a weak form of the standard *single-crossing condition* (SCC) in auction theory (Milgrom, 2004); hence the name WSCC.[1] Intuitively, WSCC means that the bidder's payment is more negatively associated with the competitor's type in X than in Y. Second, X yields at least as high interim expected revenues than Y under the reference belief (*Reference Revenue Condition, RRC*). In applications, the second condition automatically holds as an equality by the Revenue Equivalence Principle (Myerson, 1981). Thus, Theorem 2 shows that auctions with stronger negative association between a bidder' s payment and her competitor's type yield higher worst-case revenues.

The intuition of Theorem 2 is as follows. By WSCC, a bidder's payment is greater (or smaller, resp.) in X than in Y against a competitor with low types (or high types, resp.). However, a likelihood ratio dominated belief overweights low types and underweights high types. Hence, it overweights the event that the bidder's payment is greater in X than in Y, and underweights the opposite event. This, together with RRC, implies that under any decreasingly arranged belief, X yields higher interim expected revenues than Y. By Theorem 1, the worst-case revenue—the minimum expected revenue over likelihood ratio dominated beliefs—must also be higher in X than in Y.

Then, applying Theorem 2, we establish the worst-case revenue rankings between four commonly studied auctions (Theorem 3 and Figures 5-6). We find that when the reference belief is independent and identically distributed (IID) and the bidders are ambiguity neutral, (i) the first-price auction outperforms the second-price and all-pay auctions; and, (ii) the second-price and all-pay auctions outperform the war of attrition. The ranking between the second-price and all-pay auctions is indeterminate (Figure 7).

[1] Like SCC, WSCC requires that each type of bidder's transfer function in X crosses that in Y at most once, and from above. Unlike, SCC, WSCC allows the two transfer functions to touch outside the crossing point.

Notably, the worst-case revenue rankings in Theorem 3 are opposite to the expected revenue rankings in the affiliated values setup (Milgrom and Weber, 1982; Krishna and Morgan, 1997). This is because Theorem 2 works in the opposite direction to the Linkage Principle (Proposition 3). Recall that Theorem 2 states that if a bidder's payment is more negatively associated with the competitor's type in auction X than in auction Y (WSCC), then X outperforms Y. In contrast, the Linkage Principle states that if a bidder's payment is more negatively associated with her own type in X than in Y (Linkage Condition, LC; Theorem 4), then Y outperforms X. However, a negative association between a bidder's payment and her competitor's type creates a negative association between her payment and her own type in the affiliated values setup. As a result, WSCC and LC hold simultaneously, and thus the two principles predict opposite results. This logic also implies that in the presence of both ambiguity and affiliation, the rankings between the four auctions are indeterminate (Figure 8).

Our paper is related to Che and Gale (1998) in that a version of the single-crossing condition determines the revenue ranking between two auctions. To be specific, Che and Gale (1998) study the setup where each bidder has private information about her valuation and financial constraint. They show that if two auctions' iso-bid curves in the two-dimensional space of valuation and budget satisfy a single-crossing condition, the revenues from the two auctions can be compared. Waehrer et al. (1998) also analyze the setup where the seller is risk averse, a natural benchmark for our study. They show that the first-price auction outperforms the second-price auction because the winner's payment is less variable (in the sense of second-order stochastic dominance) in the first-price auction than in the second-price auction. However, their results rely on the assumption that the loser pays nothing, which is violated in the all-pay auction or war of attrition.

This paper proceeds as follows. Section 2 presents our setup. Section 3 develops our main methodology. As an application, Section 4 compares the four commonly studied auctions. Section 5 discusses the relationship between our methodology and the Linkage Principle. Section 6 provides three extensions: more than two bidders, ambiguity averse bidders and ambiguity seeking seller. Section 7 discusses the related literature and concludes.

2. Model

2.1. *Agents and preferences*

A seller wants to sell an indivisible object to two bidders, denoted as bidders 1 and 2. The two-bidder assumption is unessential; Section 6.1 generalizes our results to the I-bidder model. Each bidder has a privately known type $\theta \in \Theta = [0,1]$ representing her valuation for the object. There is a *reference belief* P, a probability measure on Θ^2, which can be interpreted as the approximating model of the true probability law (Hansen and Sargent, 2001, 2008) or the focal point distribution (Bose et al., 2006; Bose and Daripa, 2009), as mentioned in the introduction. We assume P has a positive probability density.

The seller, being ambiguity averse, holds a set of priors $\mathcal{Q}$ about the joint type distribution, containing the reference belief P. We assume $\mathcal{Q}$ is weakly compact. As mentioned in the introduction, our paper primarily focuses on the case of ambiguity neutral bidders, in which the bidders believe that types are drawn according to the reference belief P. The case of ambiguity averse bidders is discussed in Section 6.2.

For a given auction, let $t_i(\theta, \theta')$ denote bidder i's payment when her type is θ and her competitor's type is θ'. We call $t = (t_1, t_2) : \Theta^2 \to \mathbb{R}^2_+$ the *transfer function*. Following the MMEU model (Gilboa and Schmeidler, 1989), the seller evaluates an auction by the *worst-case revenue* $\mathcal{R}(t)$, the minimum expected revenue over the set of priors:

$$\mathcal{R}(t) := \min_{Q \in \mathcal{Q}} \iint_{\Theta^2} [t_1(\theta_1, \theta_2) + t_2(\theta_2, \theta_1)] Q(d\theta_1, d\theta_2).$$

2.2. *Rearrangement invariance*

This section introduces our assumption on the set of priors $\mathcal{Q}$. To explain the relevant concepts, consider a probability space (Ω, Σ, μ). Let $\Delta(\Omega, \mu)$ be the set of all probability measures over Ω absolutely continuous with respect to μ. Define a rearrangement and rearrangement invariance as follows:

Definition 1. *Let $\mathcal{S} \subset \Delta(\Omega, \mu)$.*
*(i) We say $\nu' \in \Delta(\Omega, \mu)$ is a **rearrangement** of $\nu \in \Delta(\Omega, \mu)$ (with respect to μ) if*

$$\mu\{\omega : \frac{d\nu'}{d\mu}(\omega) \le c\} = \mu\{\omega : \frac{d\nu}{d\mu}(\omega) \le c\} \quad \textit{for all } c \ge 0.$$

*(ii) For $\mathcal{Q} \subset \mathcal{S}$, we say $\mathcal{Q}$ is **rearrangement invariant** relative to $\mathcal{S}$ (with respect to*

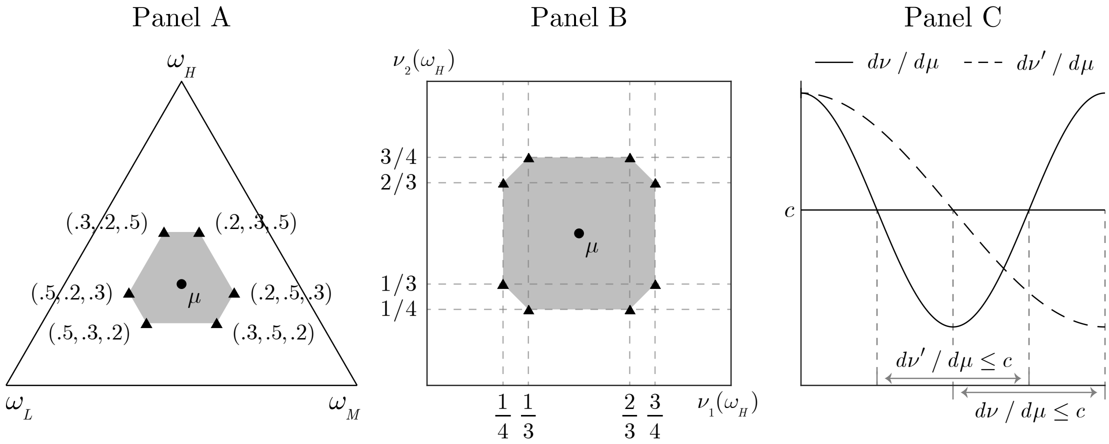

Figure 1: **Rearrangements.**
Panel A: Discrete states, the domain of all beliefs. Let $\Omega = \{\omega_L, \omega_M, \omega_H\}$ where $\omega_L < \omega_M < \omega_H$, μ be uniform on Ω, and $\mathcal{S} = \Delta(\Omega, \mu)$. A belief $\nu \in \mathcal{S}$ is represented by point $(\nu(\omega_L), \nu(\omega_M), \nu(\omega_H))$ on the simplex. Each $\nu \in \mathcal{S}$ has $3! = 6$ rearrangements, marked by triangles. The shaded hexagon is rearrangement invariant.
Panel B: Discrete states, the domain of independent beliefs. Let $\Omega = \{\omega_L, \omega_H\} \times \{\omega_L, \omega_H\}$ where $\omega_L < \omega_H$, μ be uniform on Ω, and $\mathcal{S}$ be the set of independent beliefs on Ω. A belief $\nu \in \mathcal{S}$ is represented by point $(\nu_1(\omega_H), \nu_2(\omega_H)) \in [0,1]^2$, where ν_i denotes the i-th marginal of ν. Each $\nu \in \mathcal{S}$ has $4! = 24$ rearrangements; out of these, 8 beliefs marked by triangles are independent, i.e., lie in $\mathcal{S}$. The shaded octagon is rearrangement invariant relative to $\mathcal{S}$.
Panel C: Continuous states. Let $\Omega = [0,1]$ and μ be uniform over Ω. Suppose that the Radon-Nikodym derivatives of $\nu, \nu' \in \Delta(\Omega, \mu)$ are given as in the figure. Because the lower contour sets of $d\nu/d\mu$ and $d\nu'/d\mu$ have the same total length, ν' is a rearrangement of ν.

μ) if whenever a belief belongs to $\mathcal{Q}$, its rearrangements in $\mathcal{S}$ belong to $\mathcal{Q}$: i.e.,

$$\nu \in \mathcal{Q}, \text{ and } \nu' \in \mathcal{S} \text{ is a rearrangement of } \nu \implies \nu' \in \mathcal{Q}.$$

Especially, if $\mathcal{S} = \Delta(\Omega, \mu)$, we simply say $\mathcal{Q}$ is rearrangement invariant.

Figure 1 illustrates Definition 1.

To explain Definition 1, consider the simple case of discrete Ω and uniform μ. Then, the rearrangement is equivalent to the permutation of probability masses over states (Panels A-B of Figure 1, points marked by triangles). Accordingly, rearrangement invariance requires that the set of priors, $\mathcal{Q}$, remains unchanged under permutations (Panels A-B of Figure 1, shaded regions). This means that the set of priors—and hence the degree of ambiguity it represents—is independent of the specific ordering of states. Rearrangement invariance generalizes this property to arbitrary state spaces, including continuous ones.

Rearrangement invariance in Definition 1 is also closely related to a property known as *probabilistic sophistication* (Machina and Schmeidler, 1992; Ghirardato and Marinacci, 2002; Maccheroni et al., 2006; Cerreia-Vioglio et al., 2011, 2012), also called the *neutrality axiom* (Yaari, 1987; Safra and Segal, 1998). An MMEU decision maker with set of priors $\mathcal{Q}$ is said to be probabilistically sophisticated if she is indifferent between two acts (or random payoffs) with the same outcome distribution under the reference belief: i.e., for $T, T' : \Omega \to \mathbb{R}$,

$$\mu\{\omega : T(\omega) \leq c\} = \mu\{\omega : T'(\omega) \leq c\} \quad \text{for all } c \in \mathbb{R} \tag{1}$$

$$\implies \min_{\nu \in \mathcal{Q}} \iint_{\Omega} T d\nu = \min_{\nu \in \mathcal{Q}} \iint_{\Omega} T' d\nu. \tag{2}$$

Maccheroni et al. (2006, Thm. 14) show that $\mathcal{Q}$ is rearrangement invariant if and only if the decision maker is probabilistically sophisticated. For an illustration of the "only if" implication, suppose again Ω is discrete and μ is uniform, and let $\mathcal{Q}$ be rearrangement invariant. It can be shown that if T and T' satisfy condition (1), then T' is a permutation of T. This implies that the minimum expectation of T' over $\mathcal{Q}$ can be expressed as that of T over a permutation of $\mathcal{Q}$. By rearrangement invariance, the permutation of $\mathcal{Q}$ coincides with $\mathcal{Q}$, establishing equation (2).[2] Hence, the decision maker is probabilistically sophisticated.

Now, returning to the auction setup, we consider the following three domains of beliefs $\mathcal{S}$. For a belief Q over Θ^2, let Q_i be the i-th marginal of Q.

$$\begin{aligned}
\Delta(\Theta^2, P) &:= \{Q : Q \text{ is a belief over } \Theta^2 \text{ such that } Q \ll P\} \\
\Delta^{Ind}(\Theta^2, P) &:= \{Q \in \Delta(\Theta^2, P) : Q \text{ is independent, i.e, } Q = Q_1 \times Q_2\} \\
\Delta^{IID}(\Theta^2, P) &:= \{Q \in \Delta(\Theta^2, P) : Q \text{ is IID, i.e, } Q = Q_1 \times Q_2 \text{ and } Q_1 = Q_2\}.
\end{aligned}$$

Note that $\Delta(\Theta^2, P) \supset \Delta^{Ind}(\Theta^2, P) \supset \Delta^{IID}(\Theta^2, P)$.

Our first assumption on the set of priors is given as follows:

Assumption 1A. *For $\mathcal{S} = \Delta(\Theta^2, P)$, the following holds:*
(i) $\mathcal{Q} \subset \mathcal{S}$.
(ii) $\mathcal{Q}$ is rearrangement invariant with respect to $\mathcal{S}$.

This assumption includes a wide range of sets of priors used in the robustness literature, as shown in Example 1.

Example 1 (Set of priors).
(a) Divergence neighborhood (Hansen and Sargent, 2001, 2008).

[2] More precisely, condition (1) implies that there exists a permutation σ over Ω satisfying $T' = T \circ \sigma$. Then, it can be shown that $\min_{\nu \in \mathcal{Q}} \int_{\Omega} T' d\nu = \min_{\nu \in \mathcal{Q} \circ \sigma^{-1}} \int_{\Omega} T d\nu$, where $\mathcal{Q} \circ \sigma^{-1} := \{\nu \circ \sigma^{-1} : \nu \in \mathcal{Q}\}$. Assumption 1A implies $\mathcal{Q} \circ \sigma^{-1} = \mathcal{Q}$, and hence equation (2) holds.

The ϕ-divergence is a measure of discrepancy between probability measures used in information theory and statistics (Ali and Silvey, 1966). Given a convex and continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}$, the ϕ-divergence is defined as follows: for probability measures μ and ν on the same state space,

$$D(\nu||\mu) := \int \phi\left(\frac{d\nu}{d\mu}\right) d\mu \quad \text{if } \nu \ll \mu, \quad \text{and} \quad D(\nu||\mu) := \infty \quad \text{otherwise.}$$

In the special case of $\phi(z) = z \log z$, the ϕ-divergence becomes the popular *relative entropy* (Kullback and Leibler, 1951; Kullback, 1959).

Now, let $\mathcal{Q}$ be the set of beliefs close to the reference belief, where the "closeness" is measured by the divergence:

$$\mathcal{Q} := \{Q \in \Delta(\Theta^2, P) : D(Q||P) \leq \eta\}.$$

Here, the parameter $\eta \geq 0$ represents the degree of ambiguity. Maccheroni et al. (2006, Thm. 14 and Lem. 15) show that $\mathcal{Q}$ satisfies Assumption 1A. This is one of the most popular ambiguity sets in the robustness literature (Hansen and Sargent, 2001, 2008; Ben-Tal et al., 2013).

(b) Bounded likelihood ratio (Lo, 1998; Bose et al., 2006).
Lo (1998) considers the set consisting of beliefs whose likelihood ratios lie in a given interval:

$$\mathcal{Q} := \{Q \in \Delta(\Theta^2, P) : dQ/dP \in [1 - \alpha\eta, 1 + \beta\eta]\},$$

where $\eta \geq 0$ represents the degree of ambiguity and $\alpha, \beta \geq 0$. By construction, $\mathcal{Q}$ satisfies Assumption 1A. In the limiting case of $\beta = \infty$, this model reduces to the *contamination model* (where α is normalized to 1):

$$\mathcal{Q} := \{Q = \eta R + (1 - \eta)P : R \in \Delta(\Theta^2, P)\}.$$

This model is widely used in the literature on mechanism design with ambiguity (Bose et al., 2006; Bose and Daripa, 2009). □

Some studies consider the set of priors consisting only of independent beliefs or of IID beliefs (Lo, 1998; Bose et al., 2006). This corresponds to situations where the seller has additional information that types are independent or IID. In these cases, because a rearrangement of an independent (or IID, resp.) belief is not necessarily independent (or IID, resp.), Assumption 1A does not hold. To

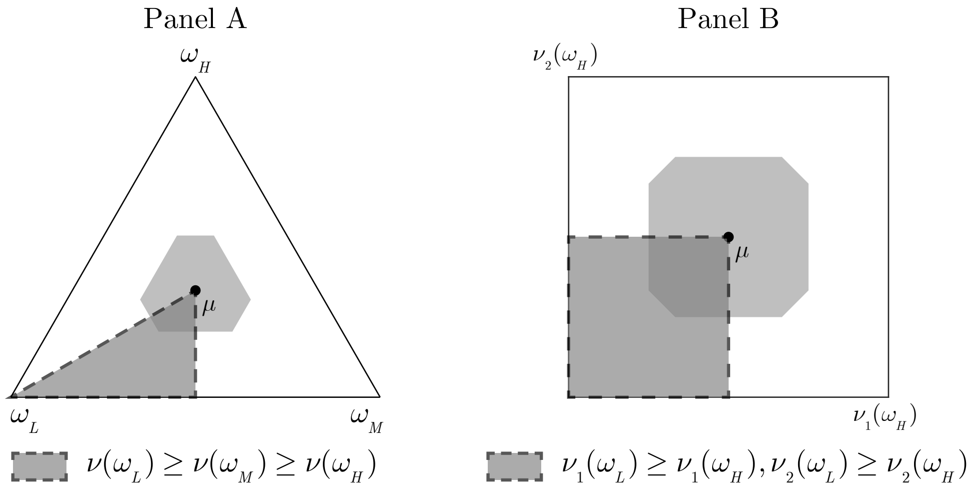

Figure 2: **Likelihood ratio dominated beliefs.** Let Ω and μ be given as in Panels A-B of Figure 1. The regions enclosed by dashed lines represent beliefs likelihood ratio dominated by μ. The intersection between two shaded regions in each panel corresponds to $\mathcal{Q}^*$ in Theorem 1.

address this issue, we suppose that the set of priors is rearrangement invariant relative to the domain of independent beliefs, or to the domain of IID beliefs.

Assumption 1B. *For $\mathcal{S} = \Delta^{Ind}(\Theta^2, P)$ or $\Delta^{IID}(\Theta^2, P)$, the following holds:*
(i) $\mathcal{Q} \subset \mathcal{S}$.
(ii) $\mathcal{Q}$ is rearrrangement invariant relative to $\mathcal{S}$.

We provide two examples satisfying Assumption 1B.

Example 2 (Set of priors: Continued)**.**
The natural analogues of Example 1 (a) and (b) are given as follows:

$$\begin{aligned}
&\textbf{(a-Ind)} \quad \mathcal{Q} := \{Q \in \Delta^{Ind}(\Theta^2, P) : D(Q_i||P_i) \leq \eta \quad \text{for } i = 1, 2\} \\
&\textbf{(a-IID)} \quad \mathcal{Q} := \{Q \in \Delta^{IID}(\Theta^2, P) : D(Q_1||P_1) \leq \eta\} \\
&\textbf{(b-Ind)} \quad \mathcal{Q} := \{Q \in \Delta^{Ind}(\Theta^2, P) : dQ_i/dP_i \in [1 - \alpha\eta, 1 + \beta\eta] \quad \text{for } i = 1, 2\} \\
&\textbf{(b-IID)} \quad \mathcal{Q} := \{Q \in \Delta^{IID}(\Theta^2, P) : dQ_1/dP_1 \in [1 - \alpha\eta, 1 + \beta\eta]\}. \quad \square
\end{aligned}$$

3. Main results

This section develops a methodology for comparing the worst-case revenues. Assumption 2 below is a common property of most standard auctions:

Assumption 2. *The total transfer $t_1(\theta, \theta') + t_2(\theta', \theta)$ increases in each argument.*

Theorem 1 states that in evaluating the worst-case revenue—the minimum expected revenue over $\mathcal{Q}$—we can restrict our attention to beliefs likelihood ratio dominated by the reference belief (Shaked and Shanthikumar, 1994).

Theorem 1. *Suppose $\mathcal{Q}$ satisfies Assumption 1A or 1B. Define $\mathcal{Q}^*$ as follows:*

$$\mathcal{Q}^* := \{Q^* \in \mathcal{Q} : Q^* \text{ is likelihood ratio dominated by } P,$$
$$\text{i.e., } \frac{dQ^*}{dP}(\theta, \theta') \text{ decreases in each argument}\}.$$

Then, for an auction whose transfer function t satisfies Assumption 2,

$$\mathcal{R}(t) = \min_{Q^* \in \mathcal{Q}^*} \iint_{\Theta^2} [t_1(\theta, \theta') + t_2(\theta', \theta)] Q^*(d\theta, d\theta').$$

Proof. See Section 3.1. □

Building on Theorem 1, Theorem 2 states that the seller prefers auction X to auction Y if two conditions hold. First, given i and θ, there exists a threshold type $\hat{\theta}$ such that bidder i of type θ pays a greater (or smaller, resp.) amount in X than in Y against a competitor of type $\theta' < \hat{\theta}$ (or $\theta' > \hat{\theta}$, resp.) (*Weak Single-Crossing Condition, WSCC*; Figure 3). This means that a bidder's payment is more negatively associated with the competitor's type in X than in Y. Second, under the reference belief, X yields at least as high interim expected revenues than Y (*Reference Revenue Condition, RRC*). In later applications, this condition automatically holds as an equality by the Revenue Equivalence Principle (Myerson, 1981). Thus, Theorem 2 shows that a negative association between a bidder's payment and her competitor's type increases the worst-case revenue.

Theorem 2. *Suppose $\mathcal{Q}$ satisfies Assumption 1A or 1B. Let X and Y be auctions whose transfers t^X and t^Y satisfy Assumption 2. Assume the following conditions:*
(i) ***Weak Single-Crossing Condition (WSCC).*** *For all i and θ, there exists a threshold type $\hat{\theta} \in [0, 1]$ such that*

$$\theta' < \hat{\theta} \implies t_i^X(\theta, \theta') \geq t_i^Y(\theta, \theta'), \quad \text{and} \quad \theta' > \hat{\theta} \implies t_i^X(\theta, \theta') \leq t_i^Y(\theta, \theta').$$

(ii) ***Reference Revenue Condition (RRC).*** *For all $i \neq j$ and θ,*

$$\int_\Theta t_i^X(\theta, \theta') P(d\theta' | \theta) \geq \int_\Theta t_i^Y(\theta, \theta') P(d\theta' | \theta),$$

where $P(\cdot | \theta)$ is the conditional distribution of bidder j's type given bidder i's type θ.[3]

[3] For notational simplicity, we omit the dependence of the conditional distribution on (i, j).

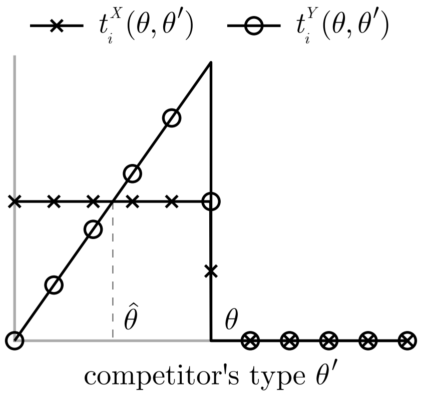

Figure 3: **WSCC (X: first-price auction / Y: second-price auction).** The "x"-ed and circled lines represent the transfer functions of bidder i with type θ in X and Y, respectively. The horizontal axis represents the competitor's type θ'. Because $t_i^X(\theta,\theta')$ lies weakly above $t_i^Y(\theta,\theta')$ for $\theta' < \hat{\theta}$ and the opposite holds for $\theta' > \hat{\theta}$, the pair (X,Y) satisfies WSCC.

Then,

$$\mathcal{R}(t^X) \geq \mathcal{R}(t^Y).$$

Proof. See Appendix B. □

We explain the intuition of Theorem 2 as follows. To prove the theorem, we establish the following inequality: for all $Q^* \in \mathcal{Q}^*$, i and θ,

$$\int_{\Theta} t_i^X(\theta,\theta')Q^*(d\theta'|\theta) \geq \int_{\Theta} t_i^Y(\theta,\theta')Q^*(d\theta'|\theta), \tag{3}$$

which implies

$$\min_{Q^* \in \mathcal{Q}^*} \iint_{\Theta^2} [t_1^X(\theta,\theta') + t_2^X(\theta',\theta)]dQ^* \geq \min_{Q^* \in \mathcal{Q}^*} \iint_{\Theta^2} [t_1^Y(\theta,\theta') + t_2^Y(\theta',\theta)]dQ^*.$$

Then, by Theorem 1, X generates a higher worst-case revenue than Y. To show inequality (3), let $Q^* \in \mathcal{Q}^*$ be given. By WSCC, bidder i of type θ pays a greater (or smaller, resp.) amount in X than in Y against a competitor with low types (or high types, resp.). However, since Q^* is likelihood ratio dominated by P, it overweights low types and underweights high types relative to P. Thus, Q^* overweights the event that the bidder's payment is greater in X than in Y, and underweights the opposite event relative to P. Because X yields at least as high interim expected revenues than Y under P by RRC, X yields higher interim expected revenues than Y under Q^*, establishing the desired inequality (3).

As mentioned in the introduction, WSCC is a weak form of the *single-crossing condition* familiar from auction theory (Milgrom, 2004, Ch. 4). Recall that t^X and

t^Y satisfy the *single-crossing condition (SCC)* if for all i, θ and $\theta' > \theta''$,

$$\begin{cases} t_i^X(\theta,\theta'') \le t_i^Y(\theta,\theta'') \implies t_i^X(\theta,\theta') \le t_i^Y(\theta,\theta') \\ t_i^X(\theta,\theta'') < t_i^Y(\theta,\theta'') \implies t_i^X(\theta,\theta') < t_i^Y(\theta,\theta'). \end{cases} \tag{4}$$

The first line means that if $t_i^X(\theta,\cdot)$ lies weakly below $t_i^Y(\theta,\cdot)$ at some point θ'', then the same holds at every higher point θ'; the second line is interpreted similarly. Now, WSCC turns out to be equivalent to the following condition (Appendix C): for all i, θ and $\theta' > \theta''$,

$$t_i^X(\theta,\theta'') < t_i^Y(\theta,\theta'') \implies t_i^X(\theta,\theta') \le t_i^Y(\theta,\theta'). \tag{5}$$

This means that if $t_i^X(\theta,\cdot)$ lies *strictly* below $t_i^Y(\theta,\cdot)$ at some point θ'', then $t_i^X(\theta,\cdot)$ lies *weakly* below $t_i^Y(\theta,\cdot)$ at every higher point θ'. It is evident that condition (5) is implied by condition (4); hence the name WSCC. Figure 3 illustrates an example that satisfies WSCC but not SCC. Like SCC, WSCC requires that $t_i^X(\theta,\cdot)$ crosses $t_i^Y(\theta,\cdot)$ at most once and from above (the point $\hat{\theta}$). However, WSCC is weaker than SCC in that it allows the two transfer functions to touch outside the crossing point (the interval $[\theta,1]$).[4]

3.1. Proof of Theorem 1

This section presents the proof of Theorem 1. We first consider the case of Assumption 1A, and then Assumption 1B.

Case A: Q satisfies Assumption 1A. The proof relies on Proposition 1, a variant of the rearrangement inequality originally due to Luxemburg (1967) and further generalized by later works.

Proposition 1 (Luxemburg, 1967, Thm. 9.1)**.**
Let $T : \Theta^2 \to \mathbb{R}_+$ be measurable and $Q \in \Delta(\Theta^2, P)$. Then,
(i) There exists a rearrangement $Q_T \in \Delta(\Theta^2, P)$ of Q such that

$$\left(\frac{dQ_T}{dP}(\theta,\theta') - \frac{dQ_T}{dP}(\varphi,\varphi')\right) \cdot (T(\theta,\theta') - T(\varphi,\varphi')) \le 0 \quad \textit{for } \theta,\theta',\varphi,\varphi' \in \Theta. \tag{6}$$

(ii) The expectation of T is smaller under Q_T than under Q:

$$\iint_{\Theta^2} T(\theta,\theta') Q_T(d\theta,d\theta') \le \iint_{\Theta^2} T(\theta,\theta') Q(d\theta,d\theta').$$

[4]Panels B and C of Figure 6 illustrate examples satisfying SCC. Also, Panel D of Figure 6 illustrates another example satisfying WSCC but not SCC.

Proof. See, e.g., Liebrich and Munari (2022), Lemmas A.1-A.2. □

A belief Q_T satisfying condition (6) is called *anti-monotone* to T; it is illustrated in Panel C of Figure 4. This condition means that dQ_T/dP and T vary in exactly opposite directions in the sense that the upper contour sets of dQ_T/dP coincide with the lower contour sets of T.[5] That is, Q_T rearranges Q in a way that assigns high (or low, resp.) probability weights to low (or high, resp.) values of T. Thus, the expectation of T is smaller under Q_T than under Q.

Then, Theorem 1 follows immediately from Proposition 1.

Proof of Theorem 1 under Assumption 1A. Let $T(\theta,\theta') := t_1(\theta,\theta') + t_2(\theta',\theta)$. For a given $Q \in \mathcal{Q}$, define Q_T as in Proposition 1 (i). Since T is increasing (Assumption 2), anti-monotonicity implies that dQ_T/dP is decreasing, i.e., $Q_T \in \mathcal{Q}^*$. Also, by Proposition 1 (ii), the expected revenue under Q^* is lower than that under Q. Thus, to evaluate the minimum expected revenue over $\mathcal{Q}$, we can restrict attention to beliefs in $\mathcal{Q}^*$.

□

Case B: $\mathcal{Q}$ satisfies Assumption 1B. In this case, Theorem 1 does not follow from Proposition 1. This is because an anti-monotone rearrangement of an independent (or IID, resp.) belief is not necessarily independent (or IID, resp.). However, Proposition 2 shows that given an independent (or IID, resp.) belief, we can construct a rearrangement satisfying similar statements to Proposition 1 (i)-(ii) while preserving the independence (or IID, resp.) property.

Proposition 2. *Let $T : \Theta^2 \to \mathbb{R}_+$ be increasing in each argument and $Q \in \Delta(\Theta^2, P)$.*
(i) If P and Q are independent, there exists an independent rearrangement Q^* such that dQ/dP decreases in each argument. Furthermore, if P and Q are IID, then Q^* is IID.
(ii) The expectation of T is smaller under Q^* than under Q:

$$\iint_{\Theta^2} T(\theta,\theta')Q^*(d\theta,d\theta') \leq \iint_{\Theta^2} T(\theta,\theta')Q(d\theta,d\theta').$$

Proof. See Appendix A. □

Panel D of Figure 4 illustrates Q^*. The intuition of Proposition 2 (ii) is as follows. Unlike the anti-monotone rearrangement Q_T, the upper contour sets of dQ^*/dP do not coincide with the lower contour sets of T. However, the upper

[5]More precisely, for a given $c \in \mathbb{R}_+$, there exists $c' \in \mathbb{R}_+$ such that $\{(\theta,\theta') : \frac{dQ_T}{dP}(\theta,\theta') \geq c\} = \{(\theta,\theta') : T(\theta,\theta') \leq c'\}$, and vice versa.

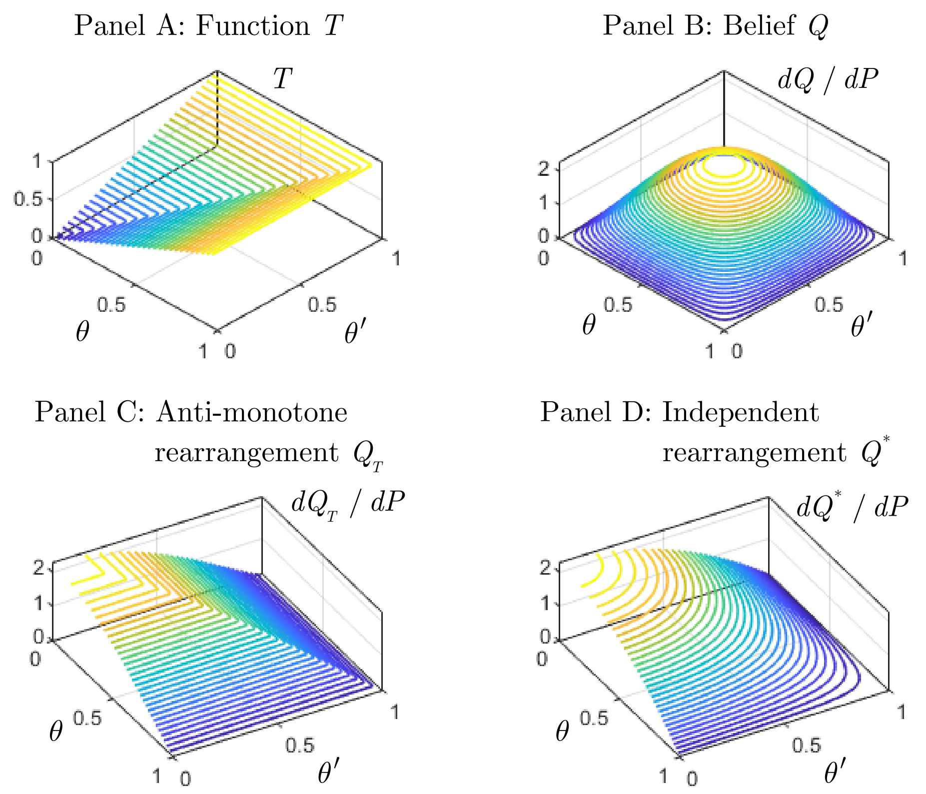

Figure 4: **Rearrangements in Proposition 1 and 2.** Let P be uniform over Θ^2. Suppose $T : \Theta^2 \to \mathbb{R}_+$ is given as in Panel A ($T(\theta, \theta') := \max\{\theta, \theta'\}$), and $Q \in \Delta(\Theta^2, P)$ as in Panel B. Then, Panel C illustrates the anti-monotone rearrangement Q_T (Proposition 1), and Panel D illustrates the independent rearrangement Q^* (Proposition 2).

contour sets of dQ^*/dP have greater intersections with the lower contour sets of T than the upper contour sets of dQ/dP have. This means that Q^* assigns greater (or smaller, resp.) probability weights to low (or high, resp.) values of T than Q does. Hence, the expectation of T is smaller under Q^* than under Q.

The proof of Theorem 1 under Assumption 1B is similar to that under Assumption 1A, and hence omitted.

4. Comparison between commonly studied auctions

In this section, assuming that the reference belief P is IID, we apply Theorem 2 to compare the worst-case revenues of four commonly studied auctions: the first-price auction (I), second-price auction (II), all-pay auction (A) and (static) war of attrition (W). For simplicity, we assume no reserve price; however, the extension to the case of reserve prices is straightforward.

Since the bidders are assumed to be ambiguity neutral, the equilibrium bidding strategies and transfer functions for the four auctions are given as follows

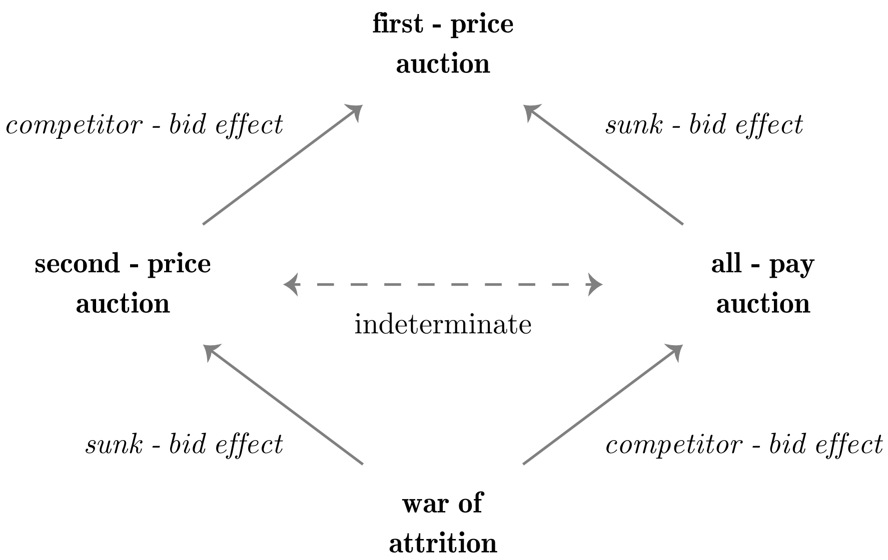

Figure 5: **Theorem 3.** Arrows indicate the direction in which the worst-case revenue increases.

(Milgrom, 2004):

$$
\begin{array}{ll}
b^I(\theta) := \theta - \int_0^\theta \frac{F(z)}{F(\theta)} dz & t_i^I(\theta, \theta') := b^I(\theta) \cdot (\mathbf{1}[\theta > \theta'] + \frac{1}{2}\mathbf{1}[\theta = \theta']) \\
b^{II}(\theta) := \theta & t_i^{II}(\theta, \theta') := b^{II}(\theta') \cdot (\mathbf{1}[\theta > \theta'] + \frac{1}{2}\mathbf{1}[\theta = \theta']) \\
b^A(\theta) := \theta F(\theta) - \int_0^\theta F(z) dz & t_i^A(\theta, \theta') := b^A(\theta) \\
b^W(\theta) := \int_0^\theta \frac{z f(z)}{1 - F(z)} dz & t_i^W(\theta, \theta') := b^W(\theta)\mathbf{1}[\theta < \theta'] + b^W(\theta')\mathbf{1}[\theta \geq \theta'],
\end{array}
$$

where $F(z) := P\{(\theta, \theta') : \theta \leq z, \theta' \in \Theta\}$ denotes the marginal cumulative distribution and $f(z) := F'(z)$ the marginal probability density.

Theorem 3, the main result of this section, establishes the worst-case revenue rankings between the four auctions (Figure 5).

Theorem 3. *Suppose $\mathcal{Q}$ satisfies Assumption 1A or 1B. If P is IID and the bidders are ambiguity neutral, the following statements hold:*
(i) $\mathcal{R}(t^I) \geq \mathcal{R}(t^{II})$.
(ii) $\mathcal{R}(t^I) \geq \mathcal{R}(t^A)$.
(iii) $\mathcal{R}(t^A) \geq \mathcal{R}(t^W)$.
(iv) Suppose that

$$\theta \times f(\theta)/[1 - F(\theta)] \text{ increases in } \theta. \tag{7}$$

Then, $\mathcal{R}(t^{II}) \geq \mathcal{R}(t^W)$.

Proof. See Appendix D. □

Condition (7) is a weak version of the usual assumption that the hazard rate $f/(1-F)$ is increasing. This condition guarantees that the equilibrium bidding strategies of the second-price auction and war of attrition, i.e., b^{II} and b^{W}, intersect exactly once (except at the origin).[6] For example, $F(\theta) \equiv \theta^{\alpha}$ satisfies condition (7), where $\alpha > 0$.

Notably, the worst-case revenue rankings between the four auctions in Theorem 3 are opposite to the expected revenue rankings in the affiliated values setup (Figure 5; Milgrom and Weber, 1982; Krishna and Morgan, 1997). Section 5 discusses the relationship between the two in more detail. Also, in the special case of the bounded likelihood ratio model (Example 2 (b-IID)), Lo (1998) shows that the first-price auction outperforms the second-price auction. Theorem 3 (i) generalizes this result.[7]

We now outline the proof of Theorem 3. By Theorem 2, to prove Theorem 3, it is sufficient to verify that the pairs $(X, Y) = (I, II), (I, A), (A, W), (II, W)$ satisfy both WSCC and RRC. Figure 6 illustrates that these pairs satisfy WSCC. Also, by the Revenue Equivalence Principle (Myerson, 1981), the four auctions yield the same interim expected revenues; hence, RRC holds as an equality. This establishes Theorem 3.

Analogously to Krishna and Morgan (1997), we identify two effects driving Theorem 3. To explain this, recall that Theorem 2 means that a negative (or positive, resp.) association between a bidder's payment and his competitor's type increases (or decreases, resp.) the worst-case revenue. First, auctions in which a bidder pays the competitor's bid underperform auctions in which a bidder pays her own bid; we name this the *competitor-bid effect* (Figure 5, arrows with upper-right directions). When a bidder pays the competitor's bid instead of her bid, a positive association between her payment and the competitor's type arises. According to Theorem 2, this positive association decreases the worst-case revenue. The competitor-bid effect explains why the second-price auction underperforms the first-price auction, and the war of attrition underperforms the all-pay auction.

Second, auctions in which bids are sunk underperform auctions in which payments are contingent on winning; we name this the *sunk-bid effect* (Figure 5, arrows with upper-left directions). The logic is similar as in the previous paragraph: the fact that a bidder pays even when she loses—in which case the

[6]Condition (7) can be weakened because Theorem 3 (iv) holds whenever b^{II} and b^{W} intersect exactly once except at the origin.

[7]Lo (1998) also analyzes the case where both the seller and bidders have MMEU preferences, with the sets of priors given by the bounded likelihood ratio model. Likewise, this result is a special case of Corollary 1 (i) in Section 6.2.

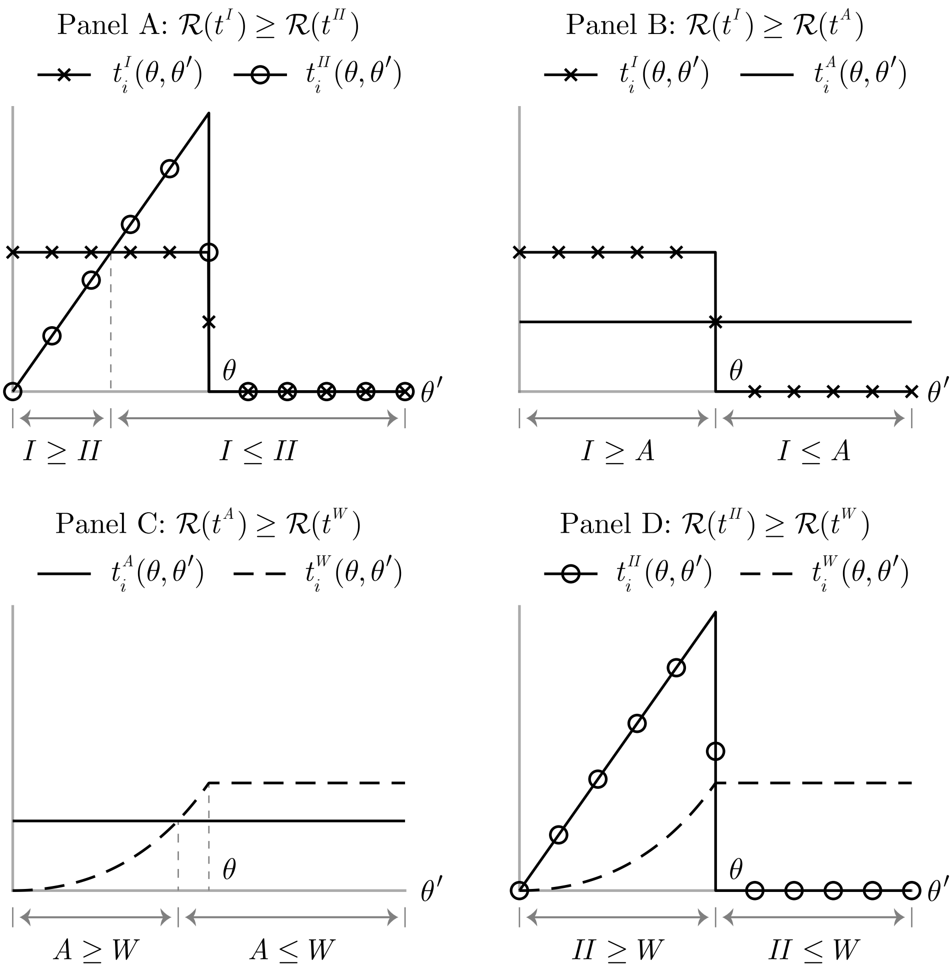

Figure 6: **Proof of Theorem 3.** Each panel plots the transfer functions of a type θ bidder in two auctions. Horizontal axes represent the competitor's type θ'.

competitor's type is high—creates a positive association between her payment and the competitor's type. The sunk-bid effect explains why the the all-pay auction underperforms first-price auction, and the second-price auction underperforms the war of attrition.

The ranking between the second-price and all-pay auctions is indeterminate, as shown in Figure 7. The reason is that whereas the competitor-bid effect makes the second-price auction inferior to the all-pay auction, the sunk-bid effect offsets this effect.

5. Relation to the Linkage Principle

As mentioned in Section 4, the worst-case revenue rankings between the four auctions in Theorem 3 are opposite to the expected revenue rankings in the affiliated values setup (Milgrom and Weber, 1982; Krishna and Morgan, 1997).

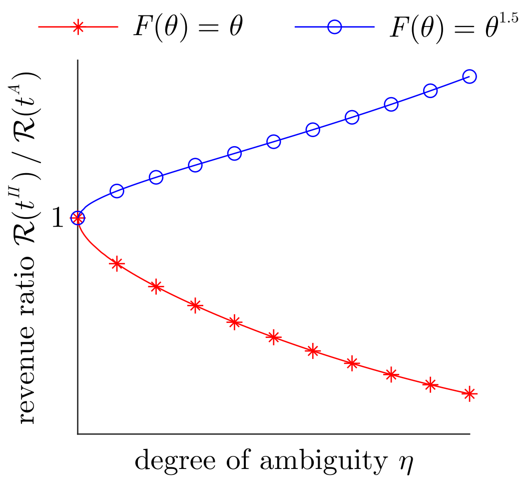

Figure 7: **Indeterminacy between II and A.** Let $\mathcal{Q}$ be the relative entropy neighborhood (Example 1 (a)): $\mathcal{Q} := \{Q \in \Delta(\Theta^2, P) : \int \log(dQ/dP)dQ \leq \eta\}$. The figure plots the revenue ratio $\mathcal{R}(t^{II})/\mathcal{R}(t^{A})$, where the horizontal axis represents the degree of ambiguity η. The starred and circled lines represent the cases where $F(\theta) = \theta$ and $F(\theta) = \theta^{1.5}$, respectively. In the former, the second-price auction outperforms the all-pay auction; in the latter, the opposite holds.

By investigating the relationship between Theorem 2 and the Linkage Principle, this section provides the intuition as to why the two results are opposite.

Let $p : \Theta^2 \to \mathbb{R}_+$ be the probability density of P. Recall that P is *symmetric* if $p(\theta, \theta') = p(\theta', \theta)$, and P is *affiliated* if

$$p(\theta_H, \theta'_L)p(\theta_L, \theta'_H) \leq p(\theta_H, \theta'_H)p(\theta_L, \theta'_L) \quad \text{for } \theta_L \leq \theta_H \text{ and } \theta'_L \leq \theta'_H.$$

Next, recall the Linkage Principle:

Theorem 4 (Linkage Principle; Krishna, 2002, Ch. 7)**.** *Suppose P is symmetric and affiliated. Let X and Y be symmetric auctions with symmetric equilibria. Denote the interim expected payments of bidder i with type θ reporting $\hat{\theta}$ as $e_i^X(\hat{\theta}, \theta)$ and $e_i^Y(\hat{\theta}, \theta)$:*

$$e_i^X(\hat{\theta}, \theta) := \int_\Theta t_i^X(\hat{\theta}, \theta')P(d\theta'|\theta) \quad \text{and} \quad e_i^Y(\hat{\theta}, \theta) := \int_\Theta t_i^Y(\hat{\theta}, \theta')P(d\theta'|\theta).$$

Suppose further that the following condition holds:
Linkage Condition (LC). *For all i and θ,*

$$\partial_2 e^X(\theta, \theta) \leq \partial_2 e^Y(\theta, \theta),$$

where ∂_2 denotes the partial derivative with respect to the second argument.
Then,

$$\int_\Theta t_i^X(\theta, \theta')P(d\theta'|\theta) \leq \int_\Theta t_i^Y(\theta, \theta')P(d\theta'|\theta).$$

According to Theorem 2, holding the reference expected revenue equal, WSCC implies that X yields a higher worst-case revenue than Y. On the other hand, according to the Linkage Principle, LC implies that Y yields a higher reference expected revenue than X. Proposition 3 below shows that between the four auctions studied in Section 4, WSCC holds if and only if LC holds. Thus, Theorem 2 and the Linkage Principle work in the opposite directions. This explains why the worst-case revenue rankings in Theorem 3 are opposite to the expected revenue rankings with affiliated values.

Proposition 3. *For $X \neq Y \in \{I, II, A, W\}$, the following conditions are equivalent:*
(i) Under any IID P, (X, Y) satisfies WSCC.
(ii) Under any symmetric and affiliated P such that X and Y have symmetric equilibria,[8] *(X, Y) satisfies LC.*

Proof. The pairs $(X, Y) = (I, II), (I, A), (A, W), (II, W), (I, W)$ satisfy both conditions (i) and (ii); the other pairs satisfy neither.

□

In words, WSCC means that a bidder's payment is more negatively associated with her competitor's type in X than in Y. On the other hand, the standard interpretation of LC is that a bidder's payment is more negatively associated with her own type in X than in Y. However, a negative association between a bidder's payment and her competitor's type creates a negative association between her payment and her own type in the affiliated value setup. As a result, WSCC and LC hold simultaneously.

A direct implication of Proposition 3 is that in the presence of both ambiguity and affiliation, the revenue rankings between the four auctions in Section 4 are indeterminate. When the effect of ambiguity dominates the effect of affiliation, the ranking is the same as in Theorem 3; in the opposite case, the ranking is the same as in Milgrom and Weber (1982) and Krishna and Morgan (1997). Figure 8 illustrates this fact by comparing the first-price and second-price auctions. Comparisons between other auction pairs yield similar results.

[8] The first-price and second-price auctions have equilibria whenever P is symmetric and affiliated. Krishna and Morgan (1997) provide sufficient conditions on P for equilibrium existence in the all-pay auction and war of attrition, omitted in our paper due to space limitation.

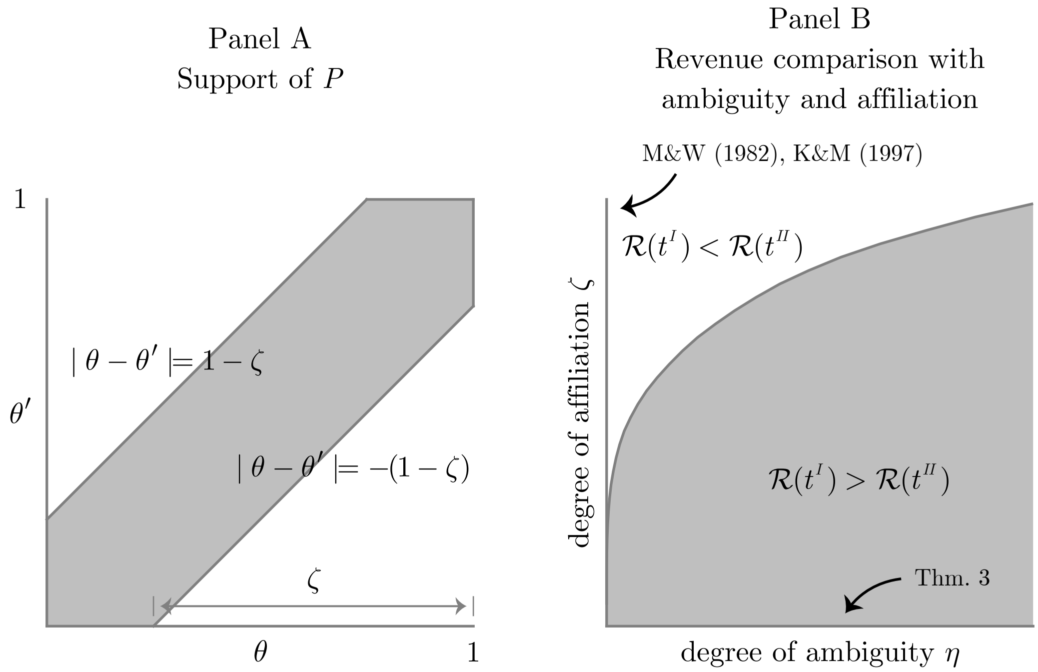

Figure 8: **Ambiguity vs. affiliation: Indeterminacy in the presence of both.**
Let $\mathcal{Q}$ be given by the relative entropy neighborhood (Example 1 (a)): $\mathcal{Q} = \{Q \in \Delta(\Theta^2, P) : \int \log(dQ/dP)dQ \leq \eta\}$. Also, let P be uniform over $\{(\theta, \theta') \in \Theta^2 : |\theta - \theta'| \leq 1 - \zeta\}$, illustrated in Panel A. If $\zeta = 0$, types are independent; if $\zeta = 1$, types are perfectly affiliated. The parameters $\eta \geq 0$ and $\zeta \in [0, 1]$ represent the degrees of ambiguity and affiliation, respectively.
Panel B compares the worst-case revenues of the first-price and second-price auctions for each (η, ζ). In the shaded region where ambiguity dominates affiliation, the ranking is the same as in Theorem 3. By contrast, in the white region where affiliation dominates ambiguity, the ranking is the same as in Milgrom and Weber (1982) and Krishna and Morgan (1997).

6. Extensions

Here, we present three extensions: more than two bidders (Section 6.1), ambiguity averse bidders (Section 6.2) and ambiguity seeking seller (Section 6.3).

6.1. *More than two bidders*

This section extends our results to the I-bidder setup. Denote bidder i's type as θ_i and the other bidders' type profile as θ_{-i}. The reference belief P is a probability measure on Θ^I, and the seller's set of priors $\mathcal{Q}$ consists of probability measures over Θ^I. As in the two-bidder model, we assume P has a positive probability density, $\mathcal{Q}$ is weakly compact, and $P \in \mathcal{Q}$. The transfer function of an auction is given as $t = (t_1, \cdots, t_I) : \Theta^I \to \mathbb{R}^I_+$, where $t_i(\theta_i, \theta_{-i})$ denotes bidder i's payment when her type is θ_i and the others' type profile is θ_{-i}. The worst-case revenue of an auction is defined as

$$\mathcal{R}(t) := \min_{Q \in \mathcal{Q}} \int_{\Theta^I} \sum_i t(\theta_i, \theta_{-i}) Q(d\theta_1, \cdots, d\theta_I).$$

We focus on auctions satisfying the following assumption:

Assumption 3. *(i) The total transfer $\sum_j t_i(\theta_j, \theta_{-j})$ increases in each argument.*
(ii) There exists $\bar{t} = (\bar{t}_1, \cdots, \bar{t}_I) : \Theta^2 \to \mathbb{R}^I_+$ such that

$$\theta_i \neq \max_{j \neq i} \theta_j \implies t_i(\theta_i, \theta_{-i}) = \bar{t}_i(\theta_i, \max_{j \neq i} \theta_j).$$

Assumption 3 (i) is analogous to Assumption 2. Assumption 3 (ii) requires that bidder i's payment can be written as a function of her own type θ_i and the highest competitor's type $\max_{j \neq i} \theta_j$ (except for the case of ties).

Theorem 5 presents the the I-bidder extension of Theorem 2. The proof is essentially the same as in Section 3, and hence omitted.

Theorem 5. *Suppose $\mathcal{Q}$ satisfies Assumption 1A or 1B. Let X and Y be auctions whose transfers t^X and t^Y satisfy Assumption 3. Assume the following conditions:*
*(i) **Weak Single-Crossing Condition (WSCC).** For all i and θ_i, there exists a threshold type $\hat{\theta} \in [0, 1]$ such that*

$$\theta' < \hat{\theta} \implies \bar{t}^X_i(\theta_i, \theta') \geq \bar{t}^Y_i(\theta_i, \theta'), \quad \textit{and} \quad \theta' > \hat{\theta} \implies \bar{t}^X_i(\theta_i, \theta') \leq \bar{t}^Y_i(\theta_i, \theta').$$

*(ii) **Reference Revenue Condition (RRC).** For all i and θ_i,*

$$\int_{\Theta^{I-1}} t^X_i(\theta_i, \theta_{-i}) P(d\theta_{-i} | \theta_i) \geq \int_{\Theta^{I-1}} t^Y_i(\theta_i, \theta_{-i}) P(d\theta_{-i} | \theta_i),$$

where $P(\cdot | \theta)$ is the conditional distribution of θ_{-i} given bidder i's type θ_i.
Then,

$$\mathcal{R}(t^X) \geq \mathcal{R}(t^Y).$$

Applying Theorem 5, it is straightforward to extend the worst-case revenue rankings in Theorem 3 to the I-bidder case.

6.2. *Ambiguity averse bidders*

Existing studies on auctions with ambiguity mostly focus on the implications of the bidders' ambiguity aversion (Table 1; see also Section 7.1). This section partially extends our results to the setup where both the sellers and the bidders exhibit ambiguity aversion. Specifically, assuming that the reference belief is IID, we show that the worst-case revenue comparison results between the four auctions remain unchanged, except for the case between the second-price auction and war of attrition.

Bidder \ Seller	Ambiguity neutral	Ambiguity averse	Ambiguity seeking
Ambiguity neutral	Myerson (1981) Milgrom and Weber (1982) Krishna and Morgan (1997)	Bose et al. (2006, Sec. 6) Sections 3-5 Section 6.1	Section 6.3
Ambiguity averse	Bose and Daripa (2009) Bodoh-Creed (2012) Laohakunakorn et al. (2019) Auster and Kellner (2022) Ghosh and Liu (2021) Baik and Hwang (2021)	Lo (1998) Bose et al. (2006, Sec. 3) Section 6.2	-

Table 1: **Comparison of the setups.** This table compares the setups studied in related works and in each section of this paper.

We represent a bidder's belief about the competitor's type by its distribution function $G : \Theta \to [0,1]$. Also, a bidder's reference belief is denoted by $F(z) := P\{(\theta, \theta') : \theta \leq z, \theta' \in \Theta\}$. Each bidder holds a set of priors $\mathcal{Q}^B$, assumed to be weakly compact. In addition, it satisfies the following assumption:

Assumption 4. *(i)* $\mathcal{Q}^B \subset \Delta(\Theta, F)$.
(ii) $\mathcal{Q}^B$ *is rearrangement invariant with respect to* $\Delta(\Theta, F)$.

Consider a symmetric sealed-bid auction in which a bidder wins the object with probability $x(b, b')$ and pays $\tau(b, b')$ when she bids b and the competitor bids b'. A bidding strategy $b^* : \Theta \to \mathbb{R}_+$ is an equilibrium if

$$b^*(\theta) \in \arg\max_b \min_{G \in \mathcal{Q}^B} \int_\Theta \left[\theta x(b, b^*(\theta')) - \tau(b, b^*(\theta'))\right] dG(\theta') \quad \text{for all } \theta.$$

Then, the transfer function is given as $t_i(\theta, \theta') := \tau(b^*(\theta), b^*(\theta'))$.

Existing studies provide closed-form formulas for the equilibria of the first-price, second-price and all-pay auctions (Lo, 1998; Baik and Hwang, 2021). Regarding the war of attrition, although an implicit characterization of the equilibrium is available, a sufficient condition for equilibrium existence is unknown. As the investigation of this problem is out of the scope of this paper, we simply assume the existence of equilibrium when necessary.

Now, we compare the worst-case revenues of the four auctions in Section 4. Because Theorem 2 imposes no restrictions on the bidders' preferences, it is applicable to the current setup. Therefore, to show that the seller prefers auction X to auction Y, it suffices to verify WSCC and RRC. Arguing as in Section 4, it is straightforward to prove that the pairs $(X, Y) = (I, II), (I, A), (A, W)$ satisfy

WSCC (provided that the war of attrition has an equilibrium). In addition, Proposition 4, proven by Baik and Hwang (2021), shows that these pairs of auctions satisfy the RRC:[9]

Proposition 4 (Baik and Hwang, 2021). *Suppose P is IID and $\mathcal{Q}^B$ satisfies Assumption 4. Then, for all i and θ,*
(i) $\int t_i^I(\theta,\theta')dF(\theta') \geq \int t_i^{II}(\theta,\theta')dF(\theta')$.
(ii) $\int t_i^I(\theta,\theta')dF(\theta') \geq \int t_i^A(\theta,\theta')dF(\theta')$.
(iii) If the war of attrition has an equilibrium, $\int t_i^A(\theta,\theta')dF(\theta') \geq \int t_i^W(\theta,\theta')dF(\theta')$.

As a result, we obtain Corollary 1, which states that Theorem 3 (i)-(iii) remain valid when the bidders are ambiguity averse.

Corollary 1. *Suppose P is IID, $\mathcal{Q}$ satisfies Assumption 1A or 1B, and $\mathcal{Q}^B$ satisfies Assumption 4. Then,*
(i) $\mathcal{R}(t^I) \geq \mathcal{R}(t^{II})$.
(ii) $\mathcal{R}(t^I) \geq \mathcal{R}(t^A)$.
(iii) If the war of attrition has an equilibrium, $\mathcal{R}(t^A) \geq \mathcal{R}(t^W)$.

The primary difficulty with extending Theorem 3 (iv), which compares the second-price auction and war of attrition, lies in the complexity of the equilibrium characterization of the war of attrition.

Remark 1. Auster and Kellner (2022) analyze the Dutch auction with ambiguity averse bidders. They find that due to dynamic inconsistency, the strategic equivalence between the Dutch and first-price auctions breaks down, and the equilibrium bidding strategy of the Dutch auction is higher than that of the first-price auction. This result, combined with Corollary 1, implies that when both the seller and the bidders exhibit ambiguity aversion, the Dutch auction outperforms the four static auctions studied in this section.

6.3. *Ambiguity seeking seller*

Experimental evidence shows that there is substantial heterogeneity in individuals' attitudes toward ambiguity, and some individuals are ambiguity seeking (Ahn et al., 2014; Chandrasekher et al., 2022). This section studies the setup

[9] Baik and Hwang's (2021) assumption on the bidders' sets of priors differs from ours. However, their proofs are valid as long as the following property holds: for any bounded measurable $\pi : \Theta \to \mathbb{R}$ and a σ-algebra $\mathcal{E}$,

$$\min_{G\in\mathcal{Q}^B}\int_\Theta \mathbb{E}_F[\pi|\mathcal{E}]d\nu \geq \min_{G\in\mathcal{Q}^B}\int_\Theta \pi d\nu.$$

Cerreia-Vioglio et al. (2012, Thm. 2) show that Assumption 4 implies this property.

where the seller displays an ambiguity seeking preference. That is, she evaluates an auction by the *best-case revenue* $\mathcal{R}^{\max}(t)$, defined as

$$\mathcal{R}^{\max}(t) := \max_{Q \in \mathcal{Q}} \iint_{\Theta^2} [t_1(\theta, \theta') + t_2(\theta', \theta)] Q(d\theta, d\theta').$$

Proposition 5 states that if auctions X and Y satisfy the opposite condition to WSCC—named the *Negative Weak Single-Crossing Condition (NWSCC)*—and RRC, then the ambiguity seeking seller prefers X to Y. The name NWSCC derives from the fact that it requires that the negatives of transfer functions satisfy WSCC: i.e., t^X and t^Y satisfy NWSCC if and only if $-t^X$ and $-t^Y$ satisfy WSCC.

Proposition 5. *Suppose $\mathcal{Q}$ satisfies Assumption 1A or 1B. Let X and Y be auctions satisfying Assumption 2. Consider the following condition:*
Negative Weak Single-Crossing Condition (NWSCC). *For all i and θ, there exists a threshold type $\hat{\theta} \in [0, 1]$ such that*

$$\theta' < \hat{\theta} \implies t_i^X(\theta, \theta') \leq t_i^Y(\theta, \theta'), \quad \textit{and} \quad \theta' > \hat{\theta} \implies t_i^X(\theta, \theta') \geq t_i^Y(\theta, \theta').$$

If (X, Y) satisfies NWSCC and RRC, then $\mathcal{R}^{\max}(t^X) \geq \mathcal{R}^{\max}(t^Y)$.

As an immediate consequence, comparisons between the four auctions yield opposite results to the ambiguity aversion case: when P is IID, (i) the war of attrition outperforms the second-price and all-pay auctions; and, (ii) the second-price and all-pay auctions outperform the first-price auction. In other words, the best-case revenue rankings between the four auctions reproduce the expected revenue rankings in the affiliated values setup (Milgrom and Weber, 1982; Krishna and Morgan, 1997).

Because the two rankings are identical, unlike the ambiguity aversion case in Section 5, the best-case revenue rankings extend to the case of symmetric and affiliated P. Specifically, using the equilibrium bidding strategies with affiliated values (Milgrom and Weber, 1982, Thm. 6 and 14; Krishna and Morgan, 1997, Thm. 1-2), it can be shown that pairs $(X, Y) = (II, I), (A, I), (W, A), (W, II)$ satisfy NWSCC. In addition, the proofs of the expected revenue rankings with affiliated values (Milgrom and Weber, 1982, Thm. 15; Krishna and Morgan, 1997, Thm. 3-5) show that the same rankings also hold for interim expected revenues; this implies that the above pairs of auctions satisfy RRC. Thus, by Proposition 5, the best-case revenue rankings remain valid in the case of symmetric and affiliated P.

7. Discussion

7.1. Related literature

Auctions with ambiguity. This paper is most closely related to the literature on auctions with ambiguity. While existing works mainly focus on the bidders' ambiguity aversion (Bose and Daripa, 2009; Bodoh-Creed, 2012; Laohakunakorn et al., 2019; Ghosh and Liu, 2021; Auster and Kellner, 2022), Bose et al. (2006, Sec. 6) show that when the seller is ambiguity averse and bidders are ambiguity neutral, the optimal mechanism is a *seller-full-insurance auction* where the total transfer is constant in the type profile. Bose et al. (2006, Sec. 3) also show that when the seller and bidders are both ambiguity averse but the seller is less averse than the bidders, the optimal mechanism is a *bidder-full-insurance auction* where a bidder's payoff is constant with respect to the competitor's type report. However, because these two mechanisms depend on bidders' beliefs, they are difficult to implement in practice and hence rarely used in reality (Wilson, 1987). We complement this result by comparing easily implementable auctions. Also, under a specific parametrization of the set of priors (Example 2 (b-IID)), Lo (1998) compares the first-price and second-price auctions. As mentioned in Section 4, our paper includes this result as a special case.

Robust auction design. Our paper is similar in spirit to the robust auction design literature (Bergemann et al., 2017, 2019; Brooks and Du, 2021; Che, 2022; He and Li, 2022; Suzdaltsev, 2022) in that the seller has limited information about the valuation distribution and evaluates auctions according to the worst-case criterion. However, our assumption on the set of probability distributions differs from this literature. Existing works consider the minimum expected revenues over (i) all information structures between valuations and signals with a given valuation distribution (Bergemann et al., 2017, 2019; Brooks and Du, 2021), (ii) all valuation distributions satisfying some moment conditions (Che, 2022; Suzdaltsev, 2022), or (iii) all correlation structures between valuations with a given marginal valuation distribution (He and Li, 2022). In contrast, our set of priors consists of beliefs close to the reference belief (Examples 1 and 2), the so-called *discrepancy-based* model (Rahimian and Mehrotra, 2019). Despite its popularity in other strands of the robustness literature—e.g., the macroeconomics literature on model misspecification (Hansen and Sargent, 2001, 2008) and the operations research literature on robust optimization (Ben-Tal et al., 2013)—the discrepancy-based model has been less frequently used in the literature on auctions where the seller has limited information about the valuation distribution.

Our paper fills this gap.

7.2. Conclusion

This paper studies the revenue comparison problem of auctions when the seller has an MMEU preference. Assuming rearrangement invariance of the set of priors, we develop a methodology for comparing the worst-case revenues of auctions. As an application, we compare the worst-case revenues of four commonly studied auctions: the first-price, second-price, all-pay auctions and war of attrition. Our methodology yields opposite results to the Linkage Principle.

Although this paper focuses on the four auctions, our methodology applies to a broader range of mechanisms. For instance, Siegel (2010) studies a mechanism in which the winner pays her bid and the loser pays a fixed fraction of her bid, called a *simple contest*. This mechanism can be regarded as a convex combination of the first-price and all-pay auctions. Applying Theorem 2, it can be shown that the worst-case revenue of a simple contest decreases in the fraction of the bid paid by the loser. In other words, the closer a simple contest is to the first-price auction (equivalently, the farther it is from the all-pay auction), the higher worst-case revenue it generates. Similar conclusions hold for the convex combinations of other auction pairs studied in Section 4.[10]

Following most of the literature on auctions with ambiguity, our paper supposes the seller has an MMEU preference. However, our results carry over to the setup where the seller has an *uncertainty averse preference*, a generalization of the MMEU preference axiomatized by Cerreia-Vioglio et al. (2011). Under rearrangement invariance assumptions analogous to Assumptions 1A-1B (see Cerreia-Vioglio et al., 2011, Sec. 4.1), it is straightforward to extend Theorems 1-3. Especially, uncertainty averse preferences include *divergence preferences* as a special case (Maccheroni et al., 2006), represented by the following functional:

$$\mathcal{R}^{\text{div}}(t) := \min_{Q \in \mathcal{Q}} \iint_{\Theta^2} [t_1(\theta_1, \theta_2) + t_2(\theta_2, \theta_1)] Q(d\theta_1, d\theta_2) + \frac{1}{\eta} D(Q||P),$$

where D is defined in Example 1 (a) and η represents the degree of ambiguity. This preference, along with the MMEU preference, is one of the most popular models in the robustness literature (Hansen and Sargent, 2001, 2008).

Appendix

[10] The proofs of these statements are available upon request.

A. Proof of Proposition 2

To prove Proposition 2, we first derive Lemma A.1 below. The proof uses the following well-known fact: if ν is an atomless probability measure on Θ and $G : \Theta \to [0,1]$ is the cumulative distribution of ν, then

$$\nu\{\theta : G(\theta) \leq c\} = c \quad \text{for } 0 \leq c \leq 1. \tag{A.1}$$

That is, if θ is distributed according to ν, then $G(\theta)$ is uniformly distributed.

Lemma A.1. *Assume P is independent. Suppose that $U \subset \Theta^2$ is an event such that*

$$\theta_L \leq \theta_H, \theta'_L \leq \theta'_H \text{ and } (\theta_L, \theta'_L) \in U \implies (\theta_H, \theta'_H) \in U. \tag{A.2}$$

Let $A_1, A_2 \subset \Theta$ be events, and $A_1^*, A_2^* \subset \Theta$ be intervals with left endpoint 0 satisfying $P_1(A_1^*) = P_1(A_1)$ and $P_2(A_2^*) = P_2(A_2)$. Then,

$$P(U \cap (A_1^* \times A_2^*)) \leq P(U \cap (A_1 \times A_2)). \tag{A.3}$$

Panel A of Figure A.9 illustrates Lemma A.1.

Proof. Let λ be the uniform measure on Θ. We proceed in three steps.
Step 1. *Without loss of generality, we can assume $P = \lambda \times \lambda$.*

For $i \in \{1,2\}$, denote the cumulative distribution of P_i as $F_i : \Theta \to [0,1]$. Let

$$\begin{aligned}
&\widehat{U} := \{(F_1(\theta), F_2(\theta')) : (\theta, \theta') \in U\} && \\
&\widehat{A}_1 := \{F_1(\theta) : \theta \in A_1\} && \widehat{A}_1^* := \{F_1(\theta) : \theta \in A_1^*\} \\
&\widehat{A}_2 := \{F_2(\theta') : \theta' \in A_2\} && \widehat{A}_2^* := \{F_2(\theta') : \theta' \in A_2^*\}.
\end{aligned}$$

By equation (A.1), if θ is distributed according to F_i, then $F_i(\theta)$ is distributed according to λ. Using this fact, it is straightforward to verify the following:
(i) $\widehat{U}$ satisfies property (A.2).
(ii) $\widehat{A}_i^*$ is an interval with left endpoint 0 satisfying $\lambda(\widehat{A}_i^*) = \lambda(\widehat{A}_i)$.
(iii) Inequality (A.3) is equivalent to

$$(\lambda \times \lambda)(\widehat{U} \cap (\widehat{A}_1^* \times \widehat{A}_2^*)) \leq (\lambda \times \lambda)(\widehat{U} \cap (\widehat{A}_1 \times \widehat{A}_2)).$$

Hence, by replacing U, A_1, A_2 and P with $\widehat{U}$, $\widehat{A}_1$, $\widehat{A}_2$ and $\lambda \times \lambda$, respectively, we can always assume $P = \lambda \times \lambda$. ■

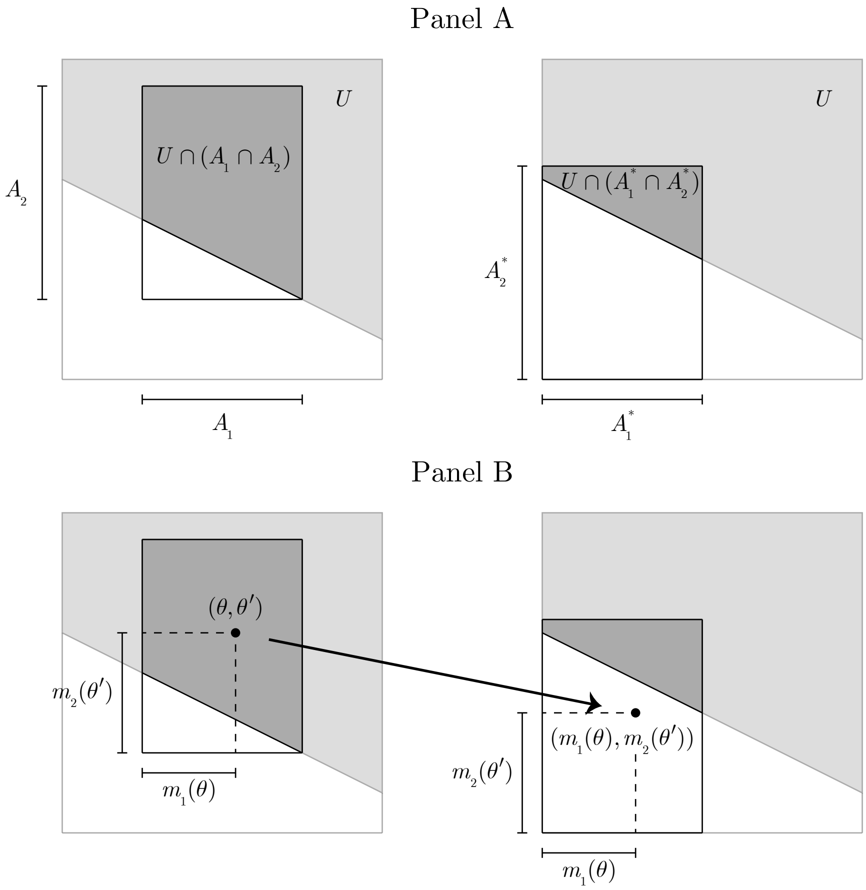

Figure A.9: **Lemma A.1.** Horizontal axes represent θ and vertical axes represent θ'. Let P be uniform over Θ^2. By assumption, A_i^* and A_i have the same total length. Panel A illustrates inequality (A.3). The intersection of $A_1^* \times A_2^*$ with U has a smaller area than that of $A_1 \times A_2$. Panel B illustrates the mapping $(\theta, \theta') \mapsto (m_1(\theta), m_2(\theta'))$ defined in Step 2 of the proof.

Step 2. *For $i \in \{1, 2\}$, define $m_i : \Theta \to A_i^*$ as*

$$m_i(\theta) := \lambda([0, \theta] \cap A_i) \quad \textit{for all } \theta. \tag{A.4}$$

Also, define measures λ_{A_i} and $\lambda_{A_i^*}$ over Θ (not necessarily probability measures) as

$$\lambda_{A_i}(E) := \lambda(E \cap A_i), \quad \textit{and} \quad \lambda_{A_i^*}(E) := \lambda(E \cap A_i^*) \quad \textit{for all event } E \subset \Theta. \tag{A.5}$$

Then,

$$\lambda_{A_i}\{\theta : m_i(\theta) \in E\} = \lambda_{A_i^*}(E) \quad \textit{for all event } E \subset \Theta.$$

Panel B of Figure A.9 illustrates the mapping $(\theta, \theta') \mapsto (m_1(\theta), m_2(\theta'))$. If $\lambda(A_i) = 0$, the proof is trivial; hence assume $\lambda(A_i) > 0$. By definition, $m_i/\lambda(A_i)$ is the cumulative distribution function of the probability measure $\lambda_{A_i}/\lambda(A_i)$

(see equations (A.4) and (A.5)). By equation (A.1), for all $c' \in [0,1]$,

$$\frac{1}{\lambda(A_i)}\lambda_{A_i}\{\theta : \frac{m_i(\theta)}{\lambda(A_i)} \le c'\} = c' \implies \lambda_{A_i}\{\theta : m_i(\theta) \le c'\lambda(A_i)\} = c'\lambda(A_i). \tag{A.6}$$

Letting $c = c'\lambda(A_i) \in [0, \lambda(A_i)]$ yields

$$\lambda_{A_i}\{\theta : m_i(\theta) \le c\} = c = \lambda_{A_i^*}([0,c]),$$

where the last equality holds because A_i^* is an interval with left endpoint 0. This establishes equation (A.6). ■

Step 3. *The desired inequality* (A.3) *holds.*

It is well-known from probability theory that definition (A.5) implies

$$\frac{d\lambda_{A_i}}{d\lambda}(\theta) = \mathbf{1}_{A_i}(\theta) \quad \text{and} \quad \frac{d\lambda_{A_i^*}}{d\lambda}(\theta) = \mathbf{1}_{A_i^*}(\theta). \tag{A.7}$$

Also, $m_i(\theta) \le \theta$ by construction. Hence, by property (A.2), $(m_1(\theta), m_2(\theta')) \in U$ implies $(\theta, \theta') \in U$. It follows that

$$\mathbf{1}_U(m_1(\theta), m_2(\theta')) \le \mathbf{1}_U(\theta, \theta'). \tag{A.8}$$

Now, we derive inequality (A.3) as follows:

$$\begin{aligned}(\lambda \times \lambda)(U \cap (A_1^* \times A_2^*)) &= \iint_{\Theta^2} \mathbf{1}_U(x,y)\lambda_{A_1^*}(dx)\lambda_{A_2^*}(dy) \\ &= \iint_{\Theta^2} \mathbf{1}_U(m_1(\theta), m_2(\theta'))\lambda_{A_1}(d\theta)\lambda_{A_2}(d\theta') \\ &\le \iint_{\Theta^2} \mathbf{1}_U(\theta,\theta')\lambda_{A_1}(d\theta)\lambda_{A_2}(d\theta') = (\lambda \times \lambda)(U \cap (A_1 \times A_2)),\end{aligned}$$

where the first and last equalities hold by equation (A.7), the second equality by Step 2 and the change of variables formula for Lebesgue integration (Shiryaev, 1996, Thm. 7 of Sec. II.6), and the third inequality by inequality (A.8).

□

Proof of Proposition 2 (i). Let $\mathbb{I} : \Theta \to \Theta$ be the identity function, i.e., $\mathbb{I}(\theta) := \theta$. Liebrich and Munari (2022, Lem. A.1-A.2) show that Proposition 1 (i) holds for arbitrary atomless probability spaces. Hence, for $i \in \{1,2\}$, there exists a rearrangement $Q_i^* \in \Delta(\Theta, P_i)$ of Q_i such that $\left(\frac{dQ_i}{dP_i}(\theta) - \frac{dQ_i}{dP_i}(\theta')\right) \cdot (\theta - \theta') \le 0$. This inequality implies that dQ_i^*/dP_i is decreasing. Now, let $Q^* := Q_1^* \times Q_2^*$. Then, Q^* is an independent rearrangement of Q such that dQ^*/dP is decreasing. Also, by construction, if P and Q are IID, then so is Q^*.

$\square$

Proof of Proposition 2 (ii). By Fubini's theorem,

$$\begin{aligned}
&\iint_{\Theta^2} T(\theta,\theta')Q(d\theta,d\theta') = \iint_{\Theta^2} T(\theta,\theta')\frac{dQ_1}{dP_1}(\theta)\frac{dQ_2}{dP_2}(\theta')P(d\theta,d\theta') \\
&= \iint_{\Theta^2}\left[\int_{\mathbb{R}^3_+} \mathbf{1}[T(\theta,\theta')>x]\mathbf{1}[\frac{dQ_1}{dP_1}(\theta)>y]\mathbf{1}[\frac{dQ_2}{dP_2}(\theta')>z]dxdydz\right]P(d\theta,d\theta') \\
&= \int_{\mathbb{R}^3_+}\left[\iint_{\Theta^2} \mathbf{1}[T(\theta,\theta')>x,\frac{dQ_1}{dP_1}(\theta)>y,\frac{dQ_2}{dP_2}(\theta')>z]P(d\theta,d\theta')\right]dxdydz \\
&= \int_{\mathbb{R}^3_+} P(\{(\theta,\theta'):T(\theta,\theta')>x\}\cap\{(\theta,\theta'):\frac{dQ_1}{dP_1}(\theta)>y,\frac{dQ_2}{dP_2}(\theta')>z\})dxdydz.
\end{aligned}$$

By the same reason,

$$\begin{aligned}
&\iint_{\Theta^2} T(\theta,\theta')Q^*(d\theta,d\theta') \\
&= \int_{\mathbb{R}^3_+} P(\{(\theta,\theta'):T(\theta,\theta')>x\}\cap\{(\theta,\theta'):\frac{dQ^*_1}{dP_1}(\theta)>y,\frac{dQ^*_2}{dP_2}(\theta')>z\})dxdydz.
\end{aligned}$$

By construction (see the proof of Proposition 2 (i)), for $i\in\{1,2\}$, Q^*_i is a rearrangement of Q_i such that dQ^*_i/dP_i is decreasing. Hence, if we let

$$\begin{array}{ll}
U := \{(\theta,\theta'):T(\theta,\theta')>x\} & \\
S_1 := \{\theta:\frac{dQ_1}{dP_1}(\theta)>y\} & S_2 := \{\theta':\frac{dQ_2}{dP_2}(\theta')>z\} \\
S^*_1 := \{\theta:\frac{dQ^*_1}{dP_1}(\theta)>y\} & S^*_2 := \{\theta':\frac{dQ^*_2}{dP_2}(\theta')>z\},
\end{array}$$

the hypothesis of Lemma A.1 holds, which implies

$$\begin{aligned}
&P(\{(\theta,\theta'):T(\theta,\theta')>x\}\cap\{(\theta,\theta'):\frac{dQ^*_1}{dP_1}(\theta)>y,\frac{dQ^*_2}{dP_2}(\theta')>z\}) \\
&\le P(\{(\theta,\theta'):T(\theta,\theta')>x\}\cap\{(\theta,\theta'):\frac{dQ_1}{dP_1}(\theta)>y,\frac{dQ_2}{dP_2}(\theta')>z\}).
\end{aligned}$$

Thus, we obtain the desired inequality:

$$\iint_{\Theta^2} T(\theta,\theta')Q^*(d\theta,d\theta') \le \iint_{\Theta^2} T(\theta,\theta')Q(d\theta,d\theta').$$

$\square$

B. Proof of Theorem 2

Let $Q^* \in \mathcal{Q}^*$ be given. As argued in Section 3, to prove Theorem 2, it suffices to prove the following: for all i and θ with $(dQ_i^*/dP_i)(\theta) > 0$,

$$\int_\Theta t_i^X(\theta, \theta')Q^*(d\theta'|\theta) \geq \int_\Theta t_i^Y(\theta, \theta')Q^*(d\theta'|\theta). \tag{B.1}$$

Note that condition $(dQ_i^*/dP_i)(\theta) > 0$ ensures that $Q^*(\cdot|\theta)$ is well-defined.

By WSCC, there exists $\hat{\theta}$ such that

$$\theta' < \hat{\theta} \implies t_i^X(\theta, \theta') \geq t_i^Y(\theta, \theta') \quad \text{and} \quad \theta' > \hat{\theta} \implies t_i^X(\theta, \theta') \leq t_i^Y(\theta, \theta').$$

Let $q^* : \Theta \to \mathbb{R}_+$ be the Radon-Nikodym derivative of $Q^*(\cdot|\theta)$ with respect to $P(\cdot|\theta)$. Since dQ^*/dP decreases in each argument, $q^*(\theta')$ decreases in θ'. Hence,

$$\begin{aligned}
\int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^+ Q^*(d\theta'|\theta) &= \int_0^{\hat{\theta}} [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^+ q^*(\theta')P(d\theta'|\theta) \\
&\geq \int_0^{\hat{\theta}} [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^+ P(d\theta'|\theta) \cdot q^*(\hat{\theta}) \\
&= \int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^+ P(d\theta'|\theta) \cdot q^*(\hat{\theta}),
\end{aligned}$$

where $z^+ := \max\{z, 0\}$. By the same reason,

$$\int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^- Q^*(d\theta'|\theta) \leq \int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^- P(d\theta'|\theta) \cdot q^*(\hat{\theta}),$$

where $z^- := \max\{-z, 0\}$. Thus,

$$\begin{aligned}
&\int_\Theta t_i^X(\theta, \theta')Q^*(d\theta'|\theta) - \int_\Theta t_i^Y(\theta, \theta')Q^*(d\theta'|\theta) \\
&= \int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^+ Q^*(d\theta'|\theta) - \int_\Theta [t_i^X(\theta, \theta') - t_i^Y(\theta, \theta')]^- Q^*(d\theta'|\theta) \\
&\geq \left[\int_\Theta t_i^X(\theta, \theta')P(d\theta'|\theta) - \int_\Theta t_i^Y(\theta, \theta')P(d\theta'|\theta) \right] \cdot q^*(\hat{\theta}) \geq 0,
\end{aligned}$$

where the last inequality holds by RRC. This establishes inequality (B.1). □

C. Weak Single-Crossing Condition

In this section, we prove the equivalence between WSCC and condition (5). To show this, it is sufficient to show Proposition C.1 below:

Proposition C.1. *Let $J, K : \Theta \to \mathbb{R}$. The following conditions are equivalent:*
(i) There exists $\hat{\theta} \in [0,1]$ such that for all θ',

$$\theta' < \hat{\theta} \implies J(\theta') \geq K(\theta'), \quad \textit{and} \quad \theta' > \hat{\theta} \implies J(\theta') \leq K(\theta').$$

(ii) For all $\theta' > \theta''$,

$$J(\theta'') < K(\theta'') \implies J(\theta') \leq K(\theta').$$

Proof. Without loss of generality, assume $K \equiv 0$.
(i) $\Rightarrow$ (ii). Suppose that $\theta' > \theta''$ and $J(\theta'') < 0$. Condition (i) implies that $\theta'' \geq \hat{\theta}$. Since $\theta' > \theta'' > \hat{\theta}$, it follows by condition (i) that $J(\theta') \leq 0$.
(ii) $\Rightarrow$ (i). We divide into two cases.
Case 1: *If $\{\theta' \in \Theta : J(\theta') < 0\} \neq \varnothing$.* In this case, define $\hat{\theta} := \inf\{\theta' \in \Theta : J(\theta') < 0\}$. Then, by definition, $J(\theta') \geq 0$ for $\theta' < \hat{\theta}$. Next, suppose $\theta' > \hat{\theta}$. By the property of the infimum, there exists $\theta'' \in [\hat{\theta}, \theta')$ such that $J(\theta'') < 0$. Condition (ii) implies that $J(\theta') \leq 0$.
Case 2: *If $\{\theta' \in \Theta : J(\theta') < 0\} = \varnothing$.* In this case, $J(\theta') \geq 0$ for all θ'. Hence, if we let $\hat{\theta} := 1$, then condition (i) holds.

□

D. Proof of Theorem 3

By Theorem 2, to prove Theorem 3, it suffices to show that the pairs $(X, Y) = (I, II), (I, A), (A, W), (II, W)$ satisfy WSCC and RRC. By the Revenue Equivalence Principle (Myerson, 1981), RRC holds. It remains to verify WSCC.
(i) $(X, Y) = (I, II)$**.** Given i and θ, let $\hat{\theta} = b^I(\theta)$. Then, since $b^I(\theta) < \theta$,

$$\begin{aligned}
\text{for } \theta' < \hat{\theta}, &\quad t_i^I(\theta, \theta') = b^I(\theta) = \hat{\theta} > \theta' = t_i^{II}(\theta, \theta') \\
\text{for } \hat{\theta} < \theta' < \theta, &\quad t_i^I(\theta, \theta') = b^I(\theta) = \hat{\theta} < \theta' = t_i^{II}(\theta, \theta') \\
\text{for } \theta' = \theta, &\quad t_i^I(\theta, \theta') = \frac{1}{2} b^I(\theta) = \frac{1}{2}\hat{\theta} < \frac{1}{2}\theta = t_i^{II}(\theta, \theta') \\
\text{for } \theta' > \theta, &\quad t_i^I(\theta, \theta') = 0 = t_i^{II}(\theta, \theta'). \quad \square
\end{aligned}$$

(ii) $(X, Y) = (I, A)$**.** Given i and θ, let $\hat{\theta} = \theta$. It is straightforward to show that

$b^I(\theta) > b^A(\theta)$. Hence,

$$\begin{aligned} \text{for } \theta' < \theta, &\quad t_i^I(\theta,\theta') = b^I(\theta) > b^A(\theta) = t_i^A(\theta,\theta') \\ \text{for } \theta' > \theta, &\quad t_i^I(\theta,\theta') = 0 < b^A(\theta) = t_i^A(\theta,\theta'). \quad \square \end{aligned}$$

(iii) $(X,Y) = (A,W)$**.** Note first that

$$\begin{aligned} b^W(\theta) &= \int_0^\theta z[-\log(1-F(z))]'dz = -\theta\log(1-F(\theta)) + \int_0^\theta \log(1-F(z))dz \\ &> \theta - \int_0^\theta F(z)dz = b^A(\theta), \end{aligned} \tag{D.1}$$

where the third inequality holds because $-\log(1-z) > z$ for $z \in (0,1)$.

Now, let i and θ be given. By inequality (D.1) and continuity, there exists $0 < \hat{\theta} < \theta$ such that $b^W(\hat{\theta}) = b^A(\theta)$. Then,

$$\begin{aligned} \text{for } \theta' < \hat{\theta}, &\quad t_i^A(\theta) = b^A(\theta) = b^W(\hat{\theta}) > b^W(\theta') = t_i^W(\theta,\theta') \\ \text{for } \hat{\theta} < \theta' < \theta, &\quad t_i^A(\theta) = b^A(\theta) = b^W(\hat{\theta}) < b^W(\theta') = t_i^W(\theta,\theta') \\ \text{for } \theta' \geq \theta, &\quad t_i^A(\theta) = b^A(\theta) < b^W(\theta) = t_i^W(\theta,\theta'). \quad \square \end{aligned}$$

(iv) $(X,Y) = (II,W)$**.** We proceed in three steps.
Step 1. $\lim_{\theta\to 0}(b^W)'(\theta) = \lim_{\theta\to 0}\theta f(\theta)/[1-F(\theta)] = 0.$

Suppose on the contrary that $\lim_{\theta\to 0}\theta f(\theta)/[1-F(\theta)] = L > 0$, where the limit exists by condition (7). Condition (7) implies further that $\theta f(\theta)/[1-F(\theta)] \geq L$. Hence, for $0 < \theta_L < \theta_H < 1$,

$$\int_{\theta_L}^{\theta_H} \frac{f(\theta)}{1-F(\theta)}d\theta \geq \int_{\theta_L}^{\theta_H} \frac{L}{\theta}d\theta \implies -\log\frac{1-F(\theta_H)}{1-F(\theta_L)} \geq L\log\frac{\theta_H}{\theta_L}.$$

Taking the limit $\theta_L \to 0$ yields $-\log[1-F(\theta_H)] \geq \infty$, a contradiction. ■
Step 2. *There exist* $\theta^* \in (0,1)$ *such that*

$$\theta \leq \theta^* \implies b^W(\theta) \leq \theta, \quad \text{and} \quad \theta \geq \theta^* \implies b^W(\theta) \geq \theta. \tag{D.2}$$

By definition, $b^W(0) = 0$. Also, by Step 1, $\lim_{\theta\to 0}(b^W)'(\theta) = 0 < 1$. It follows that for a sufficiently small θ, we have $b^W(\theta) < \theta$. Furthermore, Krishna and Morgan (1997, Prop. 1) show that $\lim_{\theta\to 1} b^W(\theta) = \infty > 1 = \lim_{\theta\to 1}\theta$. Hence, there exists an intersection $\theta^* \in (0,1)$ satisfying $b^W(\theta^*) = \theta^*$. Because condition (7) implies that $b^W(\theta) - \theta$ increases in θ, property (D.2) holds. ■

Step 3. $(X, Y) = (A, W)$ *satisfies WSCC.* Given i and θ, we divide into two cases.
Step 3-Case 1: *If* $\theta < \theta^*$. Let $\hat{\theta} = \theta$. Then,

$$\begin{aligned} \text{for } \theta' < \hat{\theta}, &\quad t_i^{II}(\theta) = \theta' > b^W(\theta') = t_i^W(\theta, \theta') \\ \text{for } \theta' > \hat{\theta}, &\quad t_i^{II}(\theta, \theta') = 0 < b^W(\theta) = t_i^W(\theta, \theta'). \end{aligned}$$

Step 3-Case 2: *If* $\theta > \theta^*$. Let $\hat{\theta} = \theta^*$. Then,

$$\begin{aligned} \text{for } \theta' < \hat{\theta}, &\quad t_i^{II}(\theta, \theta') = \theta' > b^W(\theta') = t_i^W(\theta, \theta') \\ \text{for } \hat{\theta} < \theta' < \theta, &\quad t_i^{II}(\theta, \theta') = \theta' < b^W(\theta') = t_i^W(\theta, \theta') \\ \text{for } \theta' = \theta, &\quad t_i^{II}(\theta, \theta') = \frac{1}{2}(\theta) < \theta < b^W(\theta) = t_i^W(\theta, \theta') \\ \text{for } \theta' > \theta, &\quad t_i^{II}(\theta, \theta') = 0 < b^W(\theta) = t_i^W(\theta, \theta'). \quad \square \end{aligned}$$

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