Unique stable matchings[*]

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Abstract

In this paper we show that a one-to-one two-sided matching market possesses a unique stable matching if and only if preferences on the normal form (Irving and Leather, 1986; Balinski and Ratier, 1997) are acyclic (Chung, 2000) if and only if the normal form is precisely the unique stable matching and nothing more.

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1 Introduction

In this paper we reconsider the classic model of one-to-one two-sided matching, known popularly as the *stable matching problem*. The framework, introduced in Gale and Shapley (1962), has been applied widely to settings ranging from school admissions, entry-level labour markets, refugee resettlement, and others (see Roth (2008) for a survey). Yet despite all the attention that the model has received, a classification of precisely what structural properties guarantee a unique stable matching remained a long-standing open question. In this paper we resolve this open question, providing *necessary* and *sufficient* conditions on the preferences of market participants that guarantee a unique stable matching.

Determining if a two-sided market has a unique stable matching is of practical importance. First, a unique prediction is typically viewed as a desirable property of any economic model as it saves the analyst from an "equilibrium selection" headache. Second, there is a literature, beginning with Roth (1989), that highlights the effect of incomplete information on stable matchings and uniqueness plays an important role. Third, unique stable matchings seem to appear disproportionately often in real-world matching markets, and so an understanding of their origin may help explain why.[1] Finally, uniqueness is necessary for the truthful reporting of preferences to be *strategy proof* (Gale and Sotomayor, 1985; Demange et al., 1987; Sönmez, 1999).[2] Given this, a better understanding of what assures uniqueness could potentially be levered to ensure both greater efficiency and greater transparency in practice.

While determining *if* a matching problem has a unique stable matching is important, determining *when* a matching problem has a unique stable matching is easy. Simply run the deferred acceptance algorithm of Gale and Shapley (1962) twice, once with each side in the role of proposer, and check if the stable matching found for each run is the same. If yes, then

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[1] Uniqueness has been observed in the National Resident Matching Program (Roth and Peranson, 1999), Boston school choice (Pathak and Sönmez, 2008), online dating (Hitsch et al., 2010), and the Indian marriage market (Banerjee et al., 2013).

[2] Gale and Sotomayor (1985) show that if a matching is to be generated by the deferred acceptance algorithm of Gale and Shapley (1962), then, unless there is a unique stable matching, it is always beneficial for at least one participant to misrepresent their true preferences. Example 1 in Romero-Medina and Triossi (2021) shows that a unique stable matching is not sufficient for strategy proofness. In incomplete information environments, Ehlers and Massó (2007) show that truth-telling is an ordinal Bayesian Nash equilibrium of the revelation game induced by a common belief and a stable mechanism if and only if all the profiles in the support of the common belief have singleton cores.

there is a unique stable matching; if no, then there are at least two. Since the deferred acceptance algorithm runs in polynomial time, this algorithmic approach provides an efficient solution from the perspective of computational complexity. However, while computationally efficient, the algorithmic approach sheds no light on the structure of those matching problems that possess a unique stable matching. As such, and perhaps in part also due to the problem's importance, a host of conditions on preferences that are *sufficient* for uniqueness have been proposed.[3]

Perhaps the reason a *necessary* and *sufficient* condition for uniqueness had not been found until now is that seeking conditions on the full preference lists is, in a sense to be made precise later, excessive. Rather all that matters for the set of stable matchings is how preferences operate on the essential part of the matching problem, the subproblem that we term the *normal form*.[4] Our main result, Theorem 1, shows that a unique stable matching is equivalent to preferences being *acyclic* (Chung, 2000) on the normal form which is in turn equivalent to the normal form being the unique stable matching and nothing more. The second equivalence of Theorem 1 is striking because while the normal form being the unique stable matching is trivially acyclic, the opposite implication is somewhat unexpected. Let us now explain the concepts of acyclic preferences and the normal form.

In a two-sided market with workers on one side of the market and firms on the other,[5] the shortest possible cycle involves two workers and two firms, has length 4, and is described as follows: the 1st worker prefers the 2nd firm to the 1st firm, the 2nd firm prefers the 2nd worker to the 1st worker, the 2nd worker prefers the 1st firm to the 2nd firm, and the 1st firm prefers the 1st worker to the 2nd worker. Acyclic preferences are simply those that do not possess a cycle of this kind. However, requiring that preferences are acyclic is by itself only sufficient for a unique stable matching (Romero-Medina and

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[3]Examples include the sequential preference condition (Eeckhout, 2000), the no crossing condition (Clark, 2006), the co-ranking condition (Legros and Newman, 2010), the acyclicity condition (Romero-Medina and Triossi, 2013), the universality condition (Holzman and Samet, 2014), oriented preferences (Reny, 2021), and aligned preferences (Ferdowsian et al., 2022). The concept of α-reducibility (Alcalde, 1994; Clark, 2006) is both necessary and sufficient for a matching market and any of its submatching markets to have a unique stable matching.

[4]The submatching market that is the normal form appeared first in Irving and Leather (1986) and independently in Balinski and Ratier (1997). Irving and Leather (1986) did not introduce a term to describe the normal form; Balinski and Ratier (1997) used graph-theoretic terminology and referred to it as a *domination-free marriage graph*.

[5]This will be the environment that we consider throughout. We emphasise that the terms "workers" and "firms" are merely placeholders - as mentioned in the opening paragraph the framework is applied widely.

Triossi, 2013). To add necessity, one need only require the weaker condition that preferences are acyclic on the normal form. In Example 2 we present a matching market with preferences that are not acyclic, and yet there is a unique stable matching since the preferences are acyclic on the normal form.

So what then is the normal form of a matching problem? The normal form is arrived at by stripping away parts of the preference lists that are not relevant to the set of stable matchings.[6] We term the procedure that discards the irrelevant information the *iterated deletion of unattractive alternatives* (IDUA). The IDUA procedure works by repeatedly pivoting around a particular kind of instability. Suppose that firm f is worker w's most preferred firm. Then there cannot be a stable matching that matches f to a worker that it prefers less than w, since f would propose pairing up with w and w would certainly accept. Worker w is f's outside option and we say that all the workers less preferred by f to w are *unattractive* to f. Relating this to the concept of a *reservation wage* in labour markets, worker w is in some sense the current *reservation partner* for firm f.

Since f can guarantee doing better than those unattractive workers in every stable matching, we can delete these workers from f's preference list since their presence on the list is immaterial. Similarly, all the workers deleted by f will realise that a match with f ain't happening and so will delete f from their preference lists; unattractiveness is reciprocated. The new matching environment is strictly smaller than the original environment and yet by definition the set of stable matchings can not have changed. But most importantly, further rounds of deletion may now be possible with the simplified preference lists because market participants that were not initially unattractive can become so. That is, one's outside option / reservation partner can only ever improve. Eventually the deletion procedure can go no more and what remains we call the normal form.

The IDUA procedure that reduces a matching problem to its normal form parallels closely the iterated deletion of dominated strategies (IDDS) procedure for strategic games (see Gale (1953) and Moulin (1979)). Once IDDS stops, the set of surviving strategies are the only "rational" way to play a game in which all players are rational and there is common knowledge of this fact. That is, IDDS strips a game of strategies that no rational individual who fully understands the environment could ever justify choosing; the resulting game is smaller and yet the set of solutions remains unchanged. The

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[6]Reducing a mathematical object to its bare-boned constituents, the so-called "canonical form" or "normal form", occurs not only in game theory. It is a common approach in many branches of both pure and applied mathematics (see https://en.wikipedia.org/wiki/Canonical_form).

IDUA procedure has a similar effect on matching problems. While IDUA discards information, no "important" information is discarded in the sense that it only deletes pairs that cannot be part of any stable matching; the resulting matching problem is smaller and yet the set of solutions remains unchanged.[7] In Section 2.3 we compare and contrast these two procedures with particular focus on the sorts of higher-order reasoning required for each to be operationalised.[8]

While the IDUA procedure can be justified by inferences based on higher ordering reasoning, it also admits another, more practical, interpretation that we believe is interesting. The well-known deferred acceptance algorithm of Gale and Shapley (1962) is often viewed as an interactive forum wherein those on one side of the market are assigned as active proposers and those on the other side regulated to the role of passive responders. In a similar vein, the IDUA procedure can be envisaged as a dynamic marketplace in which, perhaps more realistically, every market participant is simultaneously proposing, responding to proposals, and also handing out preemptive rejections. While Theorem 1 confirms that a matching market operating in such a fashion will not fully "clear" unless there is a unique stable matching, the market will invariably become smaller and easier to parse.[9] Interpreted in this way, IDUA has the flavour of a bargaining situation or price discovery mechanism typically more associated with traditional "markets with prices", in the sense that it closes in on the upper and lower boundaries of the (feasible region of) Pareto efficient allocations.[10]

The IDUA procedure generates ever-shortening preference lists and ever-

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[7]So from a practical perspective running IDUA should be the first port of call when considering a matching problem for which stability is a requirement. IDUA reduces the size of the input, rendering it more tractable, without affecting the desired output (the set of stable matchings). In much the same way, one often begins the analysis of a strategic game by looking for dominated strategies.

[8]Just as an ultra-sophisticated player can determine when a game is dominance solvable, so too could an ultra-sophisticated market participant determine when a matching problem has a unique stable matching.

[9]The market will only not become smaller in the statistically rare and economically unusual case that every participant's favourite partner views them as least desirable.

[10]For a more classical setting with this kind of feature, consider a seller who owns an object and values it at €5, and a buyer who values the same object at €10. In a real-world bargaining situation the buyer may start out with a bid short of €5 while the seller may first make a demand in excess of €10; only when one party crosses into the interval [€5, €10] does the "real" bargaining begin. Viewed like this, the deferred acceptance algorithm bestows upon the proposing side in a two-sided market the sort of extreme bargaining power afforded the proposer in the ultimatum game (Harsanyi, 1961; Güth et al., 1982).

shortening preference lists can be cumbersome to manage. Fortunately, the directed graph (digraph) approach to matching introduced by Maffray (1992) allows all the information of a matching problem to be encoded on a single digraph instead of multiple preference lists, allowing reductions to be handled easily. This equivalent formulation is simple to explain. Instead of "lining up" the participants on opposite sides of the market, one constructs a grid wherein each row corresponds to a specific worker, each column corresponds to a specific firm, and every vertex in the grid corresponds to a pair. Preferences are depicted by horizontal arcs (directed edges) for workers and vertical arcs for firms. A matching is depicted by a subset of vertices no two of which are in the same row nor the same column. The acyclic preferences of Chung (2000) correspond exactly to the absence of a directed cycle in the matching digraph.

The paper proceeds as follows. Section 2 introduces the matching environment that we consider, the IDUA procedure, and the normal form.[11] Section 3 presents our result classifying uniqueness. In Section 4 we introduce the digraph formulation of the matching problem. Using this setup we first illustrate how the IDUA procedure works. We then sketch how an analyst can recover, and thereby classify, the full collection of instances for which a given matching is the unique stable matching. Section 5 concludes and discusses potential avenues for future work.

All proofs appear in Appendix A. In Appendix B we show how running IDUA "in reverse" allows an analyst to identify the full collection of matching problems for which a given matching is the unique stable matching.

2 Matching problems and their normal form

In Section 2.1 we introduce the one-to-one matching environment with complete preference lists. In Section 2.2 we define what it means for market participants to be deemed unattractive. In Section 2.3 we introduce the IDUA procedure that repeatedly prunes information from preference lists that is irrelevant to the set of stable matchings of a given matching market. When the IDUA procedure stops, the resulting environment is termed the normal form.

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[11]Many of the properties of IDUA used in our proof of the main result, Theorem 1, were already obtained by Balinski and Ratier (1997) using matching digraphs. To make our paper self-contained, we prove every property that we use and whenever a result of ours is similar to one from Balinski and Ratier (1997) we explicitly state as such. Our main result, Theorem 1, is completely new.

2.1 Matching problems

Let W be a set of n workers and let F be a set of n firms. (We fix n as a positive integer at least greater than 2.) Each worker $w \in W$ has a strict preference relation, $\succ_w$, over the set of firms, and each firm $f \in F$ has a strict preference relation, $\succ_f$, over the set of workers. That is, when worker w prefers firm f' to firm f'', we will write $f' \succ_w f''$, with an analogous statement for the preferences of firms. A preference relation $\succ_w$ is said to be *complete* if all firms are in the relation; similarly for $\succ_f$. We assume throughout that all preference relations are complete.[12]

The following is the environment that we consider in this paper. It is precisely the environment originally defined in Gale and Shapley (1962).

Definition 1. An instance of the stable matching problem, P, is defined as the pair $(\{\succ_w\}_{w \in W}, \{\succ_f\}_{f \in F})$, where $\{\succ_w\}_{w \in W}$ and $\{\succ_f\}_{f \in F}$ are the collection of complete preference relations, one for each worker and firm.

A *matching* in P is a mapping μ from $W \cup F$ to itself such that: for every worker $w \in W$, $\mu(w) \in F$; for every firm $f \in F$, $\mu(f) \in W$; and for every $w, f \in W \cup F$, $\mu(w) = f$ if and only if $\mu(f) = w$.

The following is the key definition proposed by Gale and Shapley (1962).

Definition 2. Worker w and firm f form a *blocking pair* with respect to matching μ in P, if $f \succ_w \mu(w)$ and $w \succ_f \mu(f)$.

In words, (w, f) form a blocking pair with respect to matching μ if both w and f prefer each other over their partners in μ. That is, w and f would prefer to break away from their current partners and pair up together. The notion of stability is defined by the absence of a blocking pair.

Definition 3. A matching μ in P with no blocking pairs is a *stable matching*.[13]

Gale and Shapley (1962) introduced the deferred acceptance algorithm and used it to prove the following existence result.

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[12]While we have defined a preference relation as a binary relation over a set of participants, another equivalent way of defining preferences is simply to provide an ordered list where the first entry on the list is the most preferred participant, and so on. Occasionally it will be easier for us to view preferences as lists, for example in Definition 5.

[13]Recall that we have assumed that the number of workers equals the number of firms and that all preferences are complete. We do so purely for convenience. All results in the paper can be shown to hold for any one-to-one stable matching problem, with the caveat that in such instances there can exist stable matchings in which not all market participants are assured to be matched.

Theorem (Gale and Shapley (1962))**.** Every instance of the stable matching problem possesses at least one stable matching.

Example 1 below illustrates the above definitions.

Example 1. Let P_1 be an instance of the stable matching problem in which there are three workers and three firms. Precisely, $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3\}$, and preferences are as follows.

$$w_1 : f_2 \succ_{w_1} f_1 \succ_{w_1} f_3 \qquad f_1 : w_1 \succ_{f_1} w_2 \succ_{f_1} w_3$$
$$w_2 : f_2 \succ_{w_2} f_3 \succ_{w_2} f_1 \qquad f_2 : w_1 \succ_{f_2} w_2 \succ_{f_2} w_3$$
$$w_3 : f_1 \succ_{w_3} f_2 \succ_{w_3} f_3 \qquad f_3 : w_1 \succ_{f_3} w_3 \succ_{f_3} w_2$$

It can be checked that P_1 possesses two stable matchings, that we label μ_1 and μ_2. They are,

$$\begin{aligned} \mu_1 : &\quad (w_1, f_2), (w_2, f_3), (w_3, f_1) \\ \mu_2 : &\quad (w_1, f_2), (w_2, f_1), (w_3, f_3) \end{aligned} \tag{1}$$

Example 1 is straightforward. We introduce it because in the next section we use it to highlight how an instance of the stable matching problem may contain more information than is required in order to compute the full set of stable matchings for a particular market.

To give a taster of what we mean by the above, let us briefly consider Example 1 from the perspective of worker w_1 and from the perspective of firm f_2. Since w_1's most preferred firm is f_2 and f_2's most preferred worker is w_1, it must be that the pair (w_1, f_2) are in all stable matchings since otherwise (w_1, f_2) would form a blocking pair. (This is corroborated by the two stable matchings of Example 1, μ_1 and μ_2, given in (1).) That is, we observe and exploit the fact that most-preferred partners play a special role in matching problems since one half of a blocking pair is guaranteed.

But we can build on the above observation. Given that w_1 will certainly be matched with (their favourite firm) f_2 in every stable matching, the fact that w_1 has a relative preference for f_1 over f_3 is irrelevant. By this we mean the following: observe that if w_1's preferences were altered so that their relative preference for f_1 over f_3 were swapped, the set of stable matchings would remain unchanged. In fact, it is also the case that if worker w_1's preferences were incomplete, such that they preferred to be unmatched over being matched with either f_1 or f_3, the set of stable matchings would further remain unchanged. This motivates the notion of an *unattractive alternative* that is the subject of the next section.

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2.2 Unattractive alternatives

The informal discussion following Example 1 above highlights that for some instances of the matching problem the input supplied may exceed that which is required to compute the set of stable matchings. That is, since worker w_1 in Example 1 is so highly sought-after, the environment would be the same were w_1 to have incomplete preferences. But then the position in which worker w_1 appears on the preference lists of the firms with whom w_1 will not match is also irrelevant to the set of stable matchings.

We formalise this using the symbol $\sim$ to denote indifference and using the symbol ♭ as a placeholder that is "as bad" as unattractive alternatives.

Definition 4 (Unattractive alternatives). We say that

(i) firm f is an unattractive alternative to worker w, denoted ♭ $\sim_w f$, if there is some firm $f' \neq f$ such that (i) $f' \succ_w f$, and (ii) $w \succ_{f'} w'$ for all $w' \neq w$.

(ii) worker w is an unattractive alternative to firm f, denoted ♭ $\sim_f w$, if there is some worker $w' \neq w$ such that (i) $w' \succ_f w$, and (ii) $f \succ_{w'} f'$ for all $f' \neq f$.

(iii) f is an unattractive alternative to worker w whenever w is an unattractive alternative to firm f (and vice versa).

In words, condition (i) of Definition 4 says the following: worker w deems firm f unattractive if w will certainly do better than f in every stable matching. This is guaranteed when w is the most preferred worker of some firm f' that w prefers to f, because then w and f' would form a blocking pair to any matching that matches w with f. Condition (ii) of Definition 4 is the analog to condition (i) but for firms instead of workers. Condition (iii) stipulates that unattractiveness is reciprocated.

Let us now revisit Example 1 using this new terminology. We begin by considering condition (i). As noted previously, worker w_1 is firm f_2's most preferred worker. Given this the other two firms f_1 and f_3 are unattractive to w_1. It then follows that $f_2 \succ_{w_1}$ ♭ $\sim_{w_1} \{f_1, f_3\}$, where we have gathered the collection of w_1's unattractive alternatives in a set in which the order that they are listed is immaterial. By a similar reasoning, condition (ii) yields $w_1 \succ_{f_2}$ ♭ $\sim_{f_2} \{w_2, w_3\}$.

Consider now how the above statements impact instance P_1 of Example 1. It is clear that P_1 is, from the perspective of stability, identical to the

instance P_1', where P_1' is defined as, $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3\}$, and preferences are as follows:[14]

$w_1 : f_2 \succ_{w_1} ⅋$
$w_2 : f_2 \succ_{w_2} f_3 \succ_{w_2} f_1 \succ_{w_2} ⅋$
$w_3 : f_1 \succ_{w_3} f_2 \succ_{w_3} f_3 \succ_{w_3} ⅋$

$f_1 : w_1 \succ_{f_1} w_2 \succ_{f_1} w_3 \succ_{f_1} ⅋$
$f_2 : w_1 \succ_{f_2} ⅋$
$f_3 : w_1 \succ_{f_3} w_3 \succ_{f_3} w_2 \succ_{f_3} ⅋$

Let us now illustrate condition (iii) from Definition 4. Since worker w_1 has deemed both firms f_1 and f_3 unattractive, condition (iii) requires that both f_1 and f_3 reciprocate. The reason for this is that a match with w_1 is not happening for either of these firms, so maintaining w_1 in one's preference list serves no purpose. A similar statement holds for workers w_2 and w_3, both of whom reciprocate unattractiveness to firm f_2. This means that instance P_1' above is also identical to instance P_1'', where P_1'' is defined as, $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3\}$, and preferences are as follows:

$w_1 : f_2 \succ_{w_1} ⅋$
$w_2 : f_3 \succ_{w_2} f_1 \succ_{w_2} ⅋$
$w_3 : f_1 \succ_{w_3} f_3 \succ_{w_3} ⅋$

$f_1 : w_2 \succ_{f_1} w_3 \succ_{f_1} ⅋$
$f_2 : w_1 \succ_{f_2} ⅋$
$f_3 : w_3 \succ_{f_3} w_2 \succ_{f_3} ⅋$

Let us now make some observations. The first, and it is easily verified, is that the set of stable matchings for instance P_1'' coincides precisely with that of P_1. The second is that the set of stable matchings coincide despite the fact that P_1'' is, in a precise sense, strictly smaller than instance P_1. To see this we note two features: (i) for instance P_1'', every market participants' preference list is no longer than for instance P_1 (in fact each is strictly shorter), and (ii) the relative ordering of any pair in a preference list of P_1'' is the same as for P_1. Instance P_1'' contains all the relevant information of P_1 and yet is simpler to parse.

The above hints that (mutually) unattractive alternatives play no role in the set of stable matchings of any instance of the matching problem. In many ways, unattractive alternatives have much the same effect on stable matchings as strictly dominated strategies have on strategic games. We recall that deleting strictly dominated strategies from a strategic environment reduces the size of the game and yet does not change the set of rationalisable outcomes (the set of predictions). One might wonder whether deleting unattractive alternatives has the effect of reducing the input to a matching problem and yet does not affect the set of stable matchings (the set of predictions). The answer turns out to be yes.

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[14]To emphasise that instance P_1 is equivalent to this smaller instance, P_1', we omit from a participant's preference list those that are equivalent to ⅋.

Another natural question to ask is whether deleting unattractive alternatives from an instance of the matching problem can only be performed once. When deleting dominated strategies, the deletion operation can be applied on the reduced version of a game since strategies that were not initially strictly dominated can become so. The same occurs with unattractive alternatives, and the reason is that there may be participants who were not initially unattractive but are unattractive in the reduced environment. We address this in the next section wherein we define the *iterated deletion of unattractive alternatives* (IDUA), a procedure that parallels closely the iterated deletion of dominated strategies (IDDS) due to Gale (1953) and applied to voting by Moulin (1979).

2.3   IDUA and a Matching Problem's Normal Form

We now define the *iterated deletion of unattractive alternatives* (IDUA), a procedure that repeatedly prunes redundant information from the preference lists. Technically it continually deletes unattractive participants from preference lists until there remains no market participant who views any other as unattractive.

Definition 5 (The iterated deletion of unattractive alternatives (IDUA)). Given an instance of the matching problem $P = \left(\{\succ_w\}_{w \in W}, \{\succ_f\}_{f \in F} \right)$, we define $\succ_w^0 := \succ_w$ and $\succ_f^0 := \succ_f$, and for each $k \geq 1$, form the matching (sub)problem $P^k = \left(\{\succ_w^k\}_{w \in W}, \{\succ_f^k\}_{f \in F} \right)$ where for every worker w and every firm f,

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\begin{aligned}
\succ_w^k &= \left\{ f \mid f \succ_w^{k-1} \wp \text{ and } w \succ_f^{k-1} \wp \right\}, \text{ and} \\
\succ_f^k &= \left\{ w \mid w \succ_f^{k-1} \wp \text{ and } f \succ_w^{k-1} \wp \right\}.
\end{aligned}
\tag{2}
$$

Finally, define the *normal form* of matching problem P, P^*, as P^{k^*} where k^* is the minimum k such that $P^{k+1} = P^k$. That is, the normal form P^* is what remains when no further deletions are possible for some P^k. Worker preferences on the normal form are denoted $\{\succ_w^*\}$ and firm preferences by $\{\succ_f^*\}$.

The iterative part of Definition 5, given in (2), says that if worker w and firm f do not find each other mutually unattractive at some round of the iteration procedure, then neither deletes the other from their preference list during that round. That is, worker w carries firm f forward to the next round of the procedure and vice versa.

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The IDUA procedure has parallels with the IDDS procedure that are quite striking. Once the IDDS procedure stops (and it must), the set of surviving strategies are the only "rational" way to play the game. The following result, whose proof is in the Appendix, confirms a similar feature of the IDUA procedure. Precisely, it shows that while the IDUA procedure may discard information from a matching problem, no "important" information is discarded in the sense that the set of stable matchings for P can be computed using only P^*. That is, preference lists are pruned in such a way that the set of stable matchings remains unchanged.[15]

Lemma 1 (Balinski and Ratier (1997)). The iterated deletion of unattractive alternatives does not change the set of stable matchings. That is, P and its associated normal form, P^*, contain exactly the same set of stable matchings.

Let us now make two observations. The first concerns the details of the IDUA procedure that arrives at the normal form and the second pertains to the normal form's mathematical structure. First, in the discussion of unattractive alternatives following Example 1 and also in the formal statements of Definition 5, statements about deleting unattractive alternatives were made in a particular order. A natural concern then is whether the order in which the unattractive alternatives are deleted might matter. Fortunately, the first part of Lemma 2 below (that also appeared in Balinski and Ratier (1997)), whose proof is in the Appendix, shows that there is no issue with this as the resulting normal form is arrived at independently of the order in which unattractive alternatives are removed.[16]

Concerning the structure of the normal form, one might think that the following conjecture should be true (but it turns out not to be): if worker w and firm f do not find each other unattractive at any point (i.e., $f \in \{\succ_w^*\}$ and $w \in \{\succ_f^*\}$), then the pair (w, f) is contained in *some* stable matching. The second part of Lemma 2 shows that the conjecture is false. However, it is the case that if the pair (w, f) is contained in the normal form but is not part of any stable matching, then it must satisfy structural property (s).

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[15] Irving and Leather (1986) introduce an alternative deletion procedure that is discussed in detail in Section 3.2.1 of Roth and Sotomayor (1990). While the procedure is mechanical, in that it does not have a behavioural interpretation like IDUA, using the second part of Lemma 2 it can be shown that this procedure also reduces a matching problem to its normal form.

[16] An analogous result holds for IDDS when applied to finite games (see Mas-Colell et al. (1995) Exercise 8.B.4), though care must be taken with infinite games (see Dufwenberg and Stegeman (2002)).

Loosely, property (s) says that the pair must be "surrounded" by pairs that are contained in some stable matching.

Lemma 2. Let P be an instance of the matching problem. Then the normal form of P, P^*, is uniquely defined. That is, no matter in which order we repeatedly delete unattractive alternatives from preference lists, we always end up with the same P^*.

Furthermore, suppose that $f \in \{\succ_w^*\}$ and $w \in \{\succ_f^*\}$. Then, either (w, f) is part of some stable matching or the following property (s) holds.

(s): There exist firms f_{j_1} and f_{j_2} such that $f_{j_1}, f_{j_2} \in \{\succ_w^*\}$ with $f_{j_1} \succ_w^* f \succ_w^* f_{j_2}$, and there exist workers w_{i_1} and w_{i_2} such that $w_{i_1}, w_{i_2} \in \{\succ_f^*\}$ with $w_{i_1} \succ_f^* w \succ_f^* w_{i_2}$.

Given that strictly smaller instances are by definition computationally easier to handle, together Lemmas 1 and 2 imply that performing the IDUA procedure should be the first port of call for an analyst who insists on stability in a given two-sided matching market.

We now document some further connections between the IDDS procedure and the IDUA procedure. The first point of note concerns effectiveness. There are many strategic games in which IDDS does nothing. On the other hand IDUA almost always has some bite - except in the rare cases that every market participant's favourite partner ranks them last.

Like IDDS and the solution concept of *rationalizability* (Bernheim, 1984; Pearce, 1984; Tan and Werlang, 1988), IDUA can be justified by appealing to a form of "higher order reasoning" in recognising how other participants view the environment. However, the higher order reasoning invoked is different for IDUA. Both IDDS and rationalizability work by assuming "rationality" and "sophistication" on the part of individuals: rational individuals avoid dominated strategies and sophisticated individuals expect their rational opponents to do the same. And so on. With IDUA there is a "moment" at which both participants simultaneously recognise each other's unattractiveness and delete each other from their preference lists. It is then required that third parties are capable of recognising this and processing it. Third parties do so as their relative placing in the preference lists of others can have changed. While this might seem implausible at first, the solution concept of stability is coalitional in nature, so perhaps it is not unreasonable that the sort of higher order reasoning in which participants engage should be too.

Another way to highlight how the higher order reasoning differs between IDUA and IDDS can be seen by considering how each procedure works in the

first round of deletion. In a strategic game, a player can delete their own strictly dominated strategies without any knowledge of the other players' payoffs. The same is not true of IDUA in matching markets. Worker w can only decide that firm f_j is unattractive if there is some other firm, say f_k, who worker w prefers to firm f_j and for whom worker w is the most preferred worker. But for worker w to be able to do this, in addition to knowing their own preferences, importantly worker w must also have knowledge of f_k's preferences. (We note however that worker w need not know firm f_j's preferences.)

We conclude this section with an observation about the normal form of matching problems. If any two instances of the matching problem possess the same normal form, then they must have the same set of stable matchings. But one can show by example that the reverse implication does not hold. That is, it need not be the case that two instances with the same set of stable matchings have the same normal form. However, the reverse implication does hold for instances that possess the same unique stable matching. This can be exploited as follows: given a matching one can generate all instances for which that matching is the unique stable matching; simply start out with the matching in question and run all possible variants of the IDUA procedure "in reverse". Such a procedure is sketched in Appendix B.

3   Classifying unique stable matchings

Checking whether an instance of the matching problem has a unique stable matching can be done as follows. Run the deferred acceptance algorithm twice, once with workers in the role of proposers and once with firms in the role of proposers, and check if the stable matching found for each run is the same. If yes, then there is a unique stable matching. The reason for this is that the set of stable matchings form a distributive lattice of which the worker-proposing stable matching and the firm-proposing stable matching are the extreme elements (Knuth, 1996). Since the deferred acceptance algorithm runs in polynomial time, the algorithmic approach provides an efficient solution from the perspective of computational complexity.

But while computationally efficient, the algorithmic approach above sheds no light on the structure of instances that possess a unique stable matching. It is for this reason that a host of sufficient conditions on preferences ensuring uniqueness have been proposed (see the references in Footnote 3). Perhaps the reason a necessary condition had not been found before now was that, as per Lemma 1, what really matters is how preferences operate

on the normal form. It turns out that the barrier to uniqueness is preference lists that possess *cycles* (Chung, 2000) on the normal form.

Definition 6. Let $P = (\{\succ_w\}_{w \in W}, \{\succ_f\}_{f \in F})$ be an instance of the matching problem with n workers and n firms. We say that the preference lists of P possess a *cycle*, if there exists a subset of workers of size k and a subset of firms of size k (with $2 \leq k \leq n$), and an enumeration of and ordering of the participants $\{f_1, w_1, f_2, w_2, f_3, \ldots, w_{k-1}, f_k, w_k\}$ such that

$$
\begin{aligned}
f_{j+1} &\succ_{w_j} f_j & \text{for all } j = 1, \ldots, k \text{ (modulo } k), \text{ and} \\
w_j &\succ_{f_j} w_{j-1} & \text{for all } j = 1, \ldots, k \text{ (modulo } k)
\end{aligned}
\tag{3}
$$

We say that an instance P of the matching problem is *acyclic* if its preference lists do not possess a cycle.[17]

Each participant's preference list is generated by a binary relation that is antisymmetric and negatively transitive. Together these imply transitivity, which effectively translates as "individually acyclic". But while each individual preference relation is acyclic, cycles can materialise in the system as a whole due to the interconnectedness of the entire market. We defer further discussion of this until Section 4 because matching digraphs allow the representation of cycles in an intuitive way.

We now state our main result whose proof is found in Appendix A.

Theorem 1. Let P be an instance of the matching problem and let P^* be its associated normal form. Then the following three statements are equivalent.

(a) P has a unique stable matching.

(b) The normal form of P, P^*, is acyclic.

(c) In the normal form, P^*, every market participant's preference list is a singleton.

The equivalence of (a) and (b) confirms that it is cycles in preferences on the normal form that prevent uniqueness. To see why, consider a stable matching, μ, and consider a subset of market participants, S, whose preferences possess a cycle. Well it turns out that we can then "shuffle around" some participants in S, by assigning them different partners, also in S, and arrive at another stable matching. To illustrate this let us return again to

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[17]In the context of many-to-one matching environments both Ergin (2002) and Kesten (2006) provide alternative definitions of acyclic preference lists.

Example 1 that possessed two stable matchings. We restrict attention to P_1'' that is the normal form of P_1 (in that no further deletions are possible).

Let S be the subset of market participants $\{w_2, w_3, f_1, f_3\}$. Restricted to this subset, there are two stable (sub)matchings. They are $(w_2, f_1), (w_3, f_3)$ and $(w_3, f_1), (w_2, f_3)$. Note that the first of these stable (sub)matchings is firm-optimal, as evidenced by the fact that both firms are with their most preferred partner available in the normal form (while worker w_1 is the favourite worker of all firms in the original instance, P_1, worker w_1 was deleted from the preference lists of all firms bar firm f_2). Likewise, the second stable (sub)matching is worker-optimal.

When we "zoom in" further on the subset of market participants S in the normal form P_1'', we note that there is a cyclic structure to their collective preferences. To illustrate this, let us construct a sequence that begins with an arbitrarily chosen participant from this subset, and every subsequent element in the sequence is the most preferred participant of the participant listed before. As an example, if we begin the sequence with w_2, then the sequence is $(w_2, f_3, w_3, f_1, w_2, \dots)$, where the "$\dots$" indicate that the cycle has restarted. This can be formally stated as follows.

$$f_3 \succ_{w_2} f_1 \text{ and } w_3 \succ_{f_3} w_2 \text{ and } f_1 \succ_{w_3} f_3 \text{ and } w_2 \succ_{f_1} w_3 \qquad (4)$$

Now if we relabel w_2 by $w_{1'}$, f_1 by $f_{1'}$, w_3 by $w_{2'}$, and f_3 by $f_{2'}$, then the expressions in (4) read as

$$f_{2'} \succ_{w_{1'}} f_{1'} \text{ and } w_{2'} \succ_{f_{2'}} w_{1'} \text{ and } f_{1'} \succ_{w_{2'}} f_{2'} \text{ and } w_{1'} \succ_{f_{1'}} w_{2'} \qquad (5)$$

where we note that the expressions in (5) provide an example, with $k = 2$, of the condition for a cycle from Definition 6.

Romero-Medina and Triossi (2013) showed that acyclic preferences are sufficient for uniqueness. From the fact that (b) implies (a) we can see why this is true: if the preference lists are acyclic to begin with, then clearly pruning the preference lists cannot generate a cycle that was not there before. That is, if the preferences are acyclic to begin with (and hence there is a unique stable matching), then the normal form, that is by definition a (weakly) smaller matching market, must be acyclic too (and hence must possess the same unique stable matching).

Let us now consider the equivalence of parts (b) and (c). On first inspection (c) appears a far stronger condition than (b), in that clearly (c) implies (b): since for preference lists to possess a cycle at least four participants must each have a preference list of length at least 2. That (b) implies (c) means that once an acyclic (sub)matching problem is reached, the IDUA procedure

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will continue to truncate the problem until it reduces to the unique stable matching and nothing more.[18]

In the next section we formally introduce the directed graph formulation of matching problems due to Maffray (1992). While the reader not interested in the proofs of our results can skip this section, we believe that reformulating stable matching problems in this way is useful as it allows one to visualise the problem in question. To illustrate how visually intuitive this equivalent formulation is, Figure 1 depicts the normal form of instance P_1, P_1'', from Example 1. With three workers and three firms there is a 3×3 grid, where each vertex in the grid is a pair with vertex (i, j) corresponding to the pair (w_i, f_j). A matching is a subset of three vertices no two of which are in the same row nor the same column.
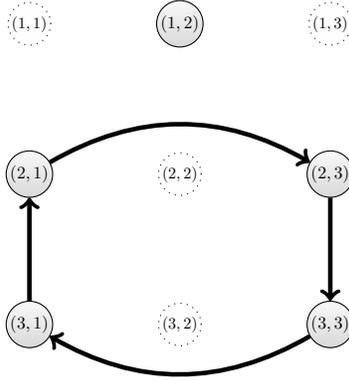

Figure 1: Illustrating a cycle in the normal form of P_1 from Example 1.

The vertices that are hollow represent pairs that cannot be part of any stable matching as both participants in that pair deemed the other unattractive at some point along the IDUA procedure. So, for example, the fact that vertex $(2, 2)$ is hollow connotes that at some point during the IDUA procedure, worker w_2 deemed firm f_2 unattractive and firm f_2 did likewise to worker w_2. The directed edges (arcs) between pairs that remain depict preferences. For example, the arc from vertex $(2, 1)$ to $(2, 3)$ indicates that worker w_2 prefers firm f_3 to firm f_1 (i.e., $f_3 \succ_{w_2} f_1$). The clockwise cycle of arcs is easily identified by simple eyeballing, and the reader can verify that

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[18]Given the similarity between IDDS for strategic games and IDUA for matching problems, matching markets with exactly one stable matching are in a sense the analog of strategic games that are *dominance solvable*.

this cycle corresponds precisely to that given in (4). (In the directed graph representing the original instance P_1, there were many more directed edges, but when a vertex in the directed graph (a pair) is deleted so too are all directed edges that are incident to it.)

4   Stable matchings and directed graphs

In Section 4.1 we introduce terminology for digraphs; the terminology is standard and so this section can be skipped by any reader familiar with digraphs. Section 4.2 introduces the equivalent matching digraph representation of a matching problem introduced by Maffray (1992) and illustrates how the IDUA procedure works via an example.

4.1   Digraph terminology and notation

A *directed graph* (or just *digraph*) D consists of a non-empty finite set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. We shall call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D and write $D = (V(D), A(D))$. For an arc xy the first vertex x is its *tail* and the second vertex y is its *head*. Moreover, x is called an *in-neighbour* of y and y an *out-neighbour* of x. We also say that the arc xy *leaves* x and *enters* y. We say that a vertex x is *incident* to an arc a if x is the head or tail of a. For a vertex $v \in V(D)$, the *out-degree* of v in D, $d_D^+(v)$, is the number of out-neighbours of v Similarly, the *in-degree* of v in D, $d_D^-(v)$, is the number of in-neighbours of v. A vertex u is *isolated* if $d_D^+(u) = d_D^-(u) = 0$.

A *walk*, W, in a digraph D is a sequence of vertices $x_1, x_2, \ldots, x_p$ for which there is an arc from each vertex in the sequence to its successor in the sequence. Such a walk is written as $W = x_1 x_2 \ldots x_p$. Special cases of walks are paths and cycles. A walk W is a *path* if the vertices of W are distinct. If the vertices $x_1, x_2, \ldots, x_{p-1}$ are distinct, for $p \geq 2$ and $x_1 = x_p$, then W is a *cycle*.

For a digraph $D = (V, A)$ and an arc $xy \in A$, *deletion of xy from D* results in the digraph $D - xy = (V, A \setminus \{xy\})$. For a vertex $v \in V$, *deletion of v from D* results in the digraph $D - v = (V \setminus \{v\}, A \setminus A_v)$, where A_v is the set of arcs in A incident to v. A digraph D' is called a *subdigraph* of D if D' is obtained from D by deleting some vertices and arcs. If only vertices are deleted, D' is an *induced subdigraph* of D.

For a textbook treatment of digraphs the reader is referred to Bang-Jensen and Gutin (2009).

4.2 Matching digraphs

Given an instance of the matching problem, P, we define the associated *matching digraph*, $D(P) = (V, A)$, where V is the vertex set and A is the arc set. The vertex set V is defined as $V := W \times F$. The arc set A is defined as follows.

$$
\begin{aligned}
A_W &:= \{(w, f_i)(w, f_j) \mid f_j \succ_w f_i\} \\
A_F &:= \{(w_i, f)(w_j, f) \mid w_j \succ_f w_i\} \\
A &:= A_W \cup A_F
\end{aligned}
$$

A matching in P is depicted in $D(P)$ by a set of vertices, M, such that for every $w \in W$ there exists exactly one vertex, (w, f'), in M containing w and for every $f \in F$ there exists exactly one vertex, (w', f), in M containing f. (Going forward we will abuse terminology and refer to such a collection of vertices in $D(P)$ as a matching.) A stable matching in $D(P)$ is a matching M such that for every vertex $(w, f) \in V(D(P))$ either $(w, f) \in M$ or (w, f) has an out-neighbour that belongs to M. In the language of directed graphs, M is a *kernel*.[19]

The following example serves three purposes. First, we use it to introduce matching digraphs in a more rigorous manner than at the end of Section 3. Second, the preference lists possess a cycle (in fact more than one), and hence none of the existing sufficient conditions available in the literature would immediately conclude that there is a unique stable matching. And yet there is a unique stable matching. Third, given that there is a unique stable matching, part (c) of Theorem 1 confirms that the IDUA will collapse each individuals preference list to a singleton. We will use this fact to illustrate how IDUA operates, and to show how visually intuitive the procedure is when a matching problem is reformulated using digraphs.

Example 2. Let P_2 be an instance of the matching problem in which there are three workers and three firms. That is, $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3\}$. Preferences are as follows:

$$
\begin{aligned}
w_1 &: f_3 \succ_{w_1} f_1 \succ_{w_1} f_2 & \qquad f_1 &: w_1 \succ_{f_1} w_2 \succ_{f_1} w_3 \\
w_2 &: f_1 \succ_{w_2} f_2 \succ_{w_2} f_3 & \qquad f_2 &: w_3 \succ_{f_2} w_1 \succ_{f_2} w_2 \\
w_3 &: f_1 \succ_{w_3} f_3 \succ_{w_3} f_2 & \qquad f_3 &: w_3 \succ_{f_3} w_2 \succ_{f_3} w_1
\end{aligned}
$$

---

[19]Kernels were first introduced in von Neumann and Morgenstern (1944) as the generalisation of solutions to cooperative games.

It can be checked that P_2 possesses a unique stable matching, μ^*, given by,

$$\mu^* : \qquad (w_1, f_1), (w_2, f_2), (w_3, f_3) \tag{6}$$

We emphasise that preferences of P_2 are not acyclic. To see this, consider the subpopulation $\{w_1, w_2, f_1, f_3\}$. The cycle here is given by,

$$f_3 \succ_{w_1} f_1 \text{ and } w_2 \succ_{f_3} w_1 \text{ and } f_1 \succ_{w_2} f_3 \text{ and } w_1 \succ_{f_1} w_2 \tag{7}$$

A relabelling of participants in (7), as was done in going from (4) to (5), confirms the cycle.

Figure 2 illustrates the matching digraph for P_2. The complete digraph $D(P_2)$ is displayed in the left hand panel, while the digraph in the right hand panel is a "condensed" version where the arcs implied by transitivity have been suppressed for readability.

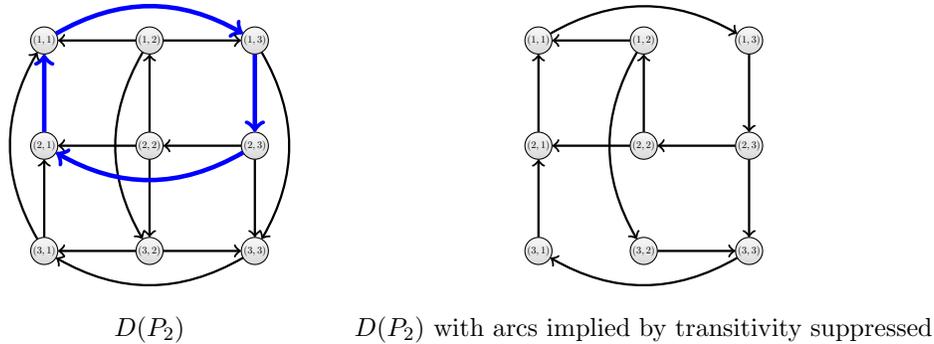$D(P_2)$ $D(P_2)$ with arcs implied by transitivity suppressed

Figure 2: The matching digraph $D(P_2)$ for the instance P_2 of Example 2.

The vertex (i, j) in each digraph of Figure 2 denotes the pair (w_i, f_j). That is, rows are indexed by workers and columns are indexed by firms. The preferences of workers are depicted by horizontal arcs, the arc set A_W, and the preferences of firms are depicted by the vertical arcs, the arc set A_F. A matching in P_2 corresponds to a set of vertices, M, no two of which are in the same row nor the same column.

Since every participant's preference is transitive, there cannot be a cycle in any row (corresponding to a worker's preferences) nor in any column (corresponding to a firm's preferences) of a matching digraph. Note however that there can be cycles in the digraph as a whole. In Figure 2, we have labelled one such preference cycle by colouring the arcs that comprise it in

blue, and we note that this is precisely the preference cycle identified in (7). So the acyclic condition due to Chung (2000), given in Definition 6, requires that preferences are intertwined in such a way that the transitivity property of individual preferences is inherited at the population level.

Given a stable matching problem P and its associated matching digraph $D(P)$, we now introduce a reduction, $\mathcal{R}$, that "prunes" the matching digraph of extraneous information. Specifically, it identifies vertices in $D(P)$ that represent worker-firm pairs that view each other as mutually unattractive and deletes them.

For every vertex $v \in V(D(P))$, it will be useful to decompose $d_D^+(v)$ into $d_D^+(v) = d_W^+(v) + d_F^+(v)$. That is, the out-degree of a vertex is split into the horizontal out-degree and the vertical out-degree. (Note that for for a given pair (w, f), the horizontal out-degree corresponds to the number of firms that w prefers to f and the vertical out-degree corresponds to the number of workers that f prefers to w.) We now have the following.

Definition 7. Given a matching digraph $D(P)$, we define $\mathcal{R}(D(P))$ as the result of the following procedure.

Choose $(w, f) \in V(D(P))$ with either $d_F^+(v) = 0$ or $d_W^+(v) = 0$, and if $d_F^+(v) = 0$ then delete all vertices (w, f_i) such that $(w, f_i)(w, f) \in A_W$. Otherwise, (i.e. $d_W^+(v) = 0$) delete all vertices (w_i, f) such that $(w_i, f)(w, f) \in A_F$.[20]

The reduction procedure, $\mathcal{R}$, operates as follows. If $d_F^+(v) = 0$, then all vertices that are the tail of an arc with head v in A_W are to be deleted. This is because if $v = (w, f)$, and $d_F^+(v) = 0$, then worker w is firm f's most preferred worker and hence w can not be matched with any firm that they prefer less than f in any stable matching. When $d_W^+(v) = 0$, analogous vertex deletions are performed.

A version of IDUA, which we call IDUA$^{\mathcal{R}}$, repeats $\mathcal{R}$ after setting $D(P) := \mathcal{R}(D(P))$ until further reductions are no longer possible. When IDUA$^{\mathcal{R}}$ stops we obtain the normal form of the initial $D(P)$ denoted by $D^*(P)$. By definition, $D^*(P) = D(P^*)$.

We emphasise that the reduction procedure IDUA$^{\mathcal{R}}$, generated by repeated applications of $\mathcal{R}$, differs slightly from the IDUA procedure of Definition 5. The difference is as follows. The way we defined IDUA in Definition 5 is more in line with the way the IDDS procedure for games is

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[20]If more than one vertex $(w, f) \in V(D(P))$ satisfies the condition above, then $\mathcal{R}(D(P))$ will depend on which of them is chosen. However, we can ignore this fact since our interest lies in repeated application of $\mathcal{R}$ (see below) and, as per Lemma 2, the end result of repeated application does not depend on the intermediate values.

typically motivated in the sense that multiple deletions happen simultaneously.[21] Specifically, in iteration k, IDUA deletes all vertices that cannot be part of some stable matching. It does this by pivoting around all blocking pairs simultaneously. This should be contrasted with the IDUA$^{\mathcal{R}}$, that in iteration k finds one blocking pair and pivots only around it. But while there is a formal difference between IDUA and IDUA$^{\mathcal{R}}$, from a practical perspective the difference is immaterial as Lemma 2 confirms that the order in which unattractive alternatives are deleted does not affect the final output.

We now illustrate how IDUA$^{\mathcal{R}}$ operates, using instance P_2 from Example 2. (Let us recall that these preferences are not acyclic and yet this instance does possess a unique stable matching.) Figure 3 below contains six panels that show repeated applications of $\mathcal{R}$ to the matching digraph of P_2, $D(P_2)$. As in Figure 2, vertex (i,j) denotes the pair (w_i, f_j). In each panel the red arrow indicates the row or column where $\mathcal{R}$ is being applied. The black vertex is the vertex with no arcs out of it in the row or column that is being considered (i.e., it is the blocking pair that $\mathcal{R}$ pivots around). The dotted vertices and arcs are the vertices and arcs that get deleted in that iteration (the dotted vertices have arcs into the black vertex, and these arcs are perpendicular to the row/column that determined the choice of the black vertex). Once the repeated application can go no further, we have the normal form of $D(P_2)$, $D^*(P_2)$, that consists of the matching in (6).

Let us now be slightly more concrete about Figure 3. In the first panel, we consider the second row that captures worker w_2's preferences. The (horizontal) arcs in this row indicate that f_1 is the most preferred firm to w_2, and so we pivot around vertex (w_2, f_1). We indicate this by colouring this vertex black. Firm f_1 recognises that worker w_2 is, in effect, a lower bound on who they can pair with. (Since if firm f_1 were matched with any worker that they prefer less than worker w_2, then (w_2, f_1) would constitute a blocking pair since w_2 will leave any firm for f_1.) Worker w_3 is one such worker. So firm f_1 deems w_3 unattractive and by the reciprocal nature of this relation, w_3 deems f_1 unattractive. Hence by considering the second row, we can delete the vertex (w_3, f_1) and any arcs incident on it.

In the second panel, consider the third row representing worker w_3's preferences. At the onset, f_1 was worker w_3's most preferred firm. But the first application of $\mathcal{R}$, depicted in the first panel, showed that this pair

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[21]While algorithmically more care must be taken with operations that delete multiple objects simultaneously, simultaneous deletion procedures are a closer fit to the higher ordering reasoning systems that game theorists assume of rational agents.
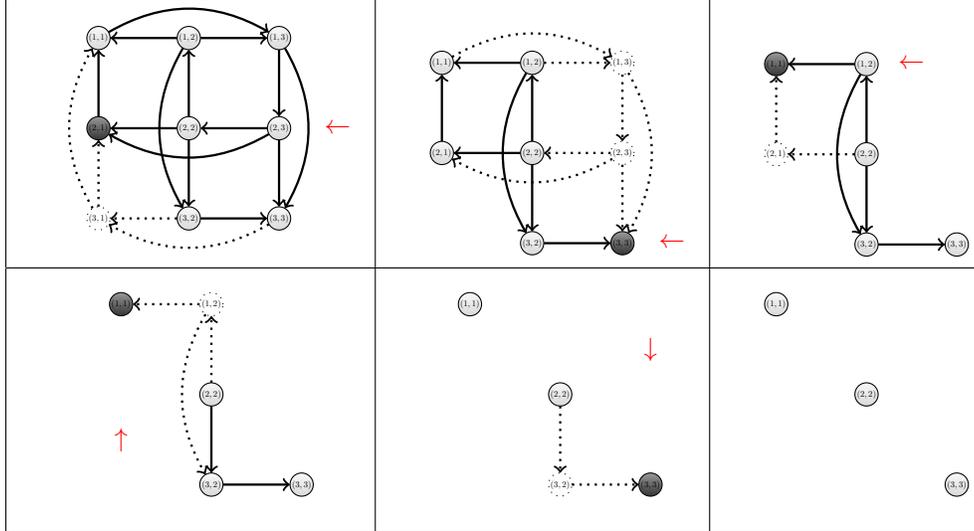
Figure 3: An illustration of how IDUA$^{\mathcal{R}}$ operates.

can never be part of a stable matching as vertex (w_3, f_1) was deleted. As such, w_3's most preferred feasible firm is f_3, which we depict by colouring the vertex (w_3, f_3) black. This means that firm f_3 can at worst match with worker w_3. Thus we can delete all vertices with arcs in A_F that have head (w_3, f_3). (In fact, given that w_3 is in fact f_3's most preferred feasible partner, we can now conclude that they will certainly be paired in all stable matchings.) Note that this deletes vertices (w_1, f_3) and (w_2, f_3), and doing so this deletes three of the four arcs comprised the cycle in the preference lists, i.e., the arcs that were coloured blue in the left hand panel of Figure 2. This highlights that cycles are not always a barrier to uniqueness.

The remaining panels should now be easily understood.

5 Conclusion and extensions for future work

Conclusion.
The two-sided matching framework of Gale and Shapley (1962) is one of the classic models of economic theory. It has been applied widely to settings ranging from school admissions, entry-level labour markets, refugee resettlement, and others (see Roth (2008) for a survey). Yet despite all the attention that the model has received, a classification of what structural

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properties guarantee a unique stable matching remained an open question. In this paper we have resolved this open question.

It turns out that the key to answering the above question is first to reduce a matching problem to its essential constituents. We do this by applying a reduction procedure, the *iterated deletion of unattractive alternatives* (IDUA), that strips away the parts of the preference lists that are not relevant to the set of stable matchings. We term the resultant (sub)matching problem the *normal form*. Our main result, Theorem 1, shows that a matching problem has a unique stable matching if and only if preferences on the normal form are acyclic if and only if the normal form is precisely the unique stable matching and nothing more (i.e., IDUA collapses every market participant's preference list to a singleton).

Extensions and open problems.
The matching environment that we have considered in this paper (Definition 1) is precisely that originally considered by Gale and Shapley (1962). This setting is defined by the following three features.

(i) *a one-to-one market*: each worker can take only one job and each firm has only one position.

(ii) *all preferences are complete*: each worker prefers to be employed at any firm over being unmatched, and each firm prefers to fill its position with any worker over leaving the position unfilled.

(iii) *a balanced market*: there are exactly the same number of workers as firms/positions.

There are many ways in which the Gale and Shapley (1962) environment has since been extended where each more general variant involves relaxing some combination of the three features above. As mentioned previously, it can be shown that the result of this paper holds if preferences are incomplete (relaxing feature (ii) above) and the market is imbalanced (relaxing feature (iii) above). The caveat is that there may be some market participants that are not included in the stable matching.

In Gutin et al. (2022) we consider many-to-one matching markets. While our focus there is not on the issue of uniqueness, using that framework it can be shown that the result of this paper extends to that of an unbalanced many-to-one matching market in which preferences need not be complete.[22]

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[22]Gutin et al. (2022) is, to the best of our knowledge, the first paper to use the digraph approach for the study of many-to-one matching environments.

This confirms that partially relaxing feature (i) is also possible. (Though again there is the caveat that not all workers will be employed and/or not all positions will be filled.) The most general environment, the so called many-to-many matching markets, present a greater challenge as there is no uniformly agreed upon notion of stability (Sotomayor, 1999; Echenique and Oviedo, 2006; Konishi and Ünver, 2006). Thus, if and how our uniqueness result carries over to many-to-many matching markets is an open problem.

Ashlagi et al. (2017) consider a variant of the matching problem with feature (iii) above relaxed. Precisely, they consider an unbalanced one-to-one market with n workers and $n + 1$ firms where all preferences are complete. Such an environment has precisely $(n!)^n((n + 1)!)^{n+1}$ possible instances. Amongst other things, Ashlagi et al. (2017) show that as n tends to infinity the fraction of instances with a unique stable matching tends to 1. They attribute this finding to a consequence of increased *competition*.

Our classification of unique stable matchings complements this result of Ashlagi et al. (2017) in the following way. If their result is a statement about *how* the core becomes small as an unbalanced market size grows, then our Theorem 1 is a statement about *why* the core becomes small as an unbalanced market size grows. That is, as the size of unbalanced markets grows, for most instances the preferences on the normal form are acyclic. This generates a puzzle. The normal form of an unbalanced $n \times (n + 1)$ matching market with complete preferences is a balanced $n \times n$ market. So how is it that an additional market participant typically induces an acyclic normal form as n gets large?

APPENDIX

A   Proofs omitted from the main text

Lemma 1, Lemma 2, and Theorem 1 are each stated in the text in terms of
the standard primitives of the model: preferences. Since all of our results
are proved using the equivalent matching digraph formulation, below we
first restate each result as it appeared in the main text and we then state
the equivalent result in terms of matching digraphs. That is, Lemma 1
is reformulated as Lemma $1'$, Lemma 2 as Lemma $2'$, and Theorem 1 as
Theorem $1'$.

Lemma 1. The iterated deletion of unattractive alternatives does not change
the set of stable matchings. That is, P and its associated normal form, P^*,
contain exactly the same set of stable matchings.

Lemma $1'$. The iterated deletion of unattractive alternatives does not
change the set of stable matchings. That is, $D(P)$ and $D^*(P)$ contain ex-
actly the same stable matchings.

Proof of Lemma $1'$ (and hence of Lemma 1).

Proof. We will show that one iteration of $\mathcal{R}$ does not change the set of stable
matchings, which will imply that the lemma holds. Assume that there is
no arc in A_F leaving $(w, f) \in V(D(P))$ and we have therefore deleted all
vertices (w, f_i) such that $(w, f_i)(w, f) \in A_W$ (see (i) in Definition 7). Let
D_1 be the matching digraph before the reduction and let D_2 denote the
matching digraph after the operation.

For the sake of contradiction assume that one of the deleted vertices, say
(w, f_i), lies in a stable matching M_1 of D_1. Then for some j, we have (w_j, f)
is also in M_1. Now we will show that (w, f) is a blocking pair in M_1. Since
(w, f_i) and (w_j, f) are in M_1, $(w, f) \notin M_1$. Moreover, f prefers w to w_j as
$(w_j, f)(w, f) \in A_F$ and w prefers f to f_i as $(w, f_i)(w, f) \in A_W$. Therefore,
M_1 does not exist (as there is a blocking pair). So no deleted vertex can
belong to a stable matching of D_1.

Therefore, if M_1 is a stable matching in D_1, then M_1 is also a stable
matching in D_2 (as D_2 is an induced subdigraph of D_1 and $M_1 \subseteq V(D_2)$).
Conversely assume that M_2 is a stable matching in D_2. Clearly no vertex
in D_2 is a blocking pair for M_2. For the sake of contradiction assume that
(w, f_i) is a blocking pair for M_2 in D_1 (where (w, f_i) is deleted when con-
structing D_2). Recall that $(w, f_i)(w, f) \in A_W$ and there are no arcs in A_F

leaving $(w, f) \in V(D(P))$. As (w, f) is not a blocking pair for M_2 in D_2, we note that (w, f) either belongs to M_2 or there is a vertex $(w, f_j) \in M_2$, where $(w, f)(w, f_j) \in A_W$. As $(w, f_i)(w, f) \in A_W$ and $(w, f)(w, f_j) \in A_W$ we note that $(w, f_i)(w, f_j) \in A_W$ and therefore (w, f_i) is not a blocking pair for M_2 in D_1. This implies that there is no blocking pair for M_2 in D_1 and therefore M_2 is a stable matching in D_1. This completes the proof. $\square$

Lemma 2. Let P be an instance of the matching problem. Then the normal form of P, P^*, is uniquely defined. That is, no matter in which order we repeatedly delete unattractive alternatives from preference lists, we always end up with the same P^*.

Furthermore, suppose that $f \in \{\succ_w^*\}$ and $w \in \{\succ_f^*\}$. Then, either (w, f) is part of some stable matching or the following property (s) holds.

(s): There exist firms f_{j_1} and f_{j_2} such that $f_{j_1}, f_{j_2} \in \{\succ_w^*\}$ with $f_{j_1} \succ_w^* f \succ_w^* f_{j_2}$, and there exist workers w_{i_1} and w_{i_2} such that $w_{i_1}, w_{i_2} \in \{\succ_f^*\}$ with $w_{i_1} \succ_f^* w \succ_f^* w_{i_2}$.

Lemma 2′. Let P be an instance of the matching problem. Then the matching digraph of the normal form, $D^*(P)$, is uniquely defined. That is, no matter in which order we repeatedly apply $\mathcal{R}$ to perform IDUA, we always end up with the same $D^*(P)$.

Furthermore, if $M \subseteq V(D(P))$ denotes the vertices that belong to some stable matching of P, then $D^*(P)$ contains a vertex (w, f) if and only if $(w, f) \in M$ or the following property (s) holds.

(s): There exists vertices $(w_{i_1}, f), (w_{i_2}, f), (w, f_{j_1}), (w, f_{j_2}) \in M$, such that $(w_{i_1}, f)(w, f)(w_{i_2}, f)$ and $(w, f_{j_1})(w, f)(w, f_{j_2})$ are paths in $V(D(P))$.

Proof of Lemma 2′ (and hence of Lemma 2).

Proof. Let P be an instance that contains a stable matching. Let $M \subseteq V(D(P))$ be the vertices of $D(P)$ that belong to some stable matching of P. Let M^* denote the set of vertices $(w, f) \in D(P)$ where either $(w, f) \in M$ or there are vertices $(w, f_i) \in M$ and $(w_j, f) \in M$ such that $(w, f_i)(w, f) \in A_W$ and $(w_j, f)(w, f) \in A_F$. We will show that no matter in what order we perform the reductions we always obtain $V(D^*(P)) = M^*$ and $A(D^*(P))$ contain the arcs of $D(P)$ with both tail and head in M^*.

Let m^* be an arbitrary vertex in M^*. If m^* would be deleted by some Reduction $\mathcal{R}$, then either all vertices incident to arcs in A_W that enter m^* will also be deleted or all vertices incident to arcs in A_F that enter m^* will be deleted (by the definition of Reduction $\mathcal{R}$). In both cases at least one vertex

from M will be deleted, a contradiction to Lemma $1'$. So, if $m^* \in M^*$ then m^* is not deleted by any Reduction $\mathcal{R}$ and $m^* \in V(D^*(P))$. This implies that $M^* \subseteq V(D^*(P))$.

For the sake of contradiction assume that there exists a pair $(w, f) \in V(D^*(P)) \setminus M^*$. As $(w, f) \notin M^*$ we note that $(w, f) \notin M$ and without loss of generality we may assume that there is no arc $a \in A_W$ such that (w, f) is the head of a and the tail of a lies in M. Since M_F (see the definition in Lemma 3) is a stable matching it contains a vertex $(w, f_i) \in M_F$. As (w, f) has no in-neighbour of the form (w, f_j) belonging to M, and therefore also not belonging to M_F, we have that (w, f_i) is an out-neighbour of (w, f). However as there are no arc in A_F leaving (w, f_i) and $(w, f)(w, f_i) \in A_W$, we note that (w, f) will be deleted by Reduction $\mathcal{R}$, a contradiction to $(w, f) \in V(D^*(P))$. So $V(D^*(P)) = M^*$ as desired.

We will now prove the second part of the lemma. If $(w, f) \in M$ or if (w, f) satisfies Property (s) then clearly $(w, f) \in M^* = V(D^*(P))$ (due to the existence of (w_{i_1}, f) and (w, f_{j_1}) if Property (s) holds).

Conversely let $(w, f) \in M^*$ be arbitrary. We want to show that $(w, f) \in M$ or that Property (s) holds. For the sake of contradiction assume that this is not the case. As $(w, f) \in M^*$ and $(w, f) \notin M$ we note that there exists $(w_{i_1}, f), (w, f_{j_1}) \in M$, such that $(w_{i_1}, f)(w, f) \in A_F$ and $(w, f_{j_1})(w, f) \in A_W$. Let $M_{(w_{i_1}, f)}$ denote the stable matching in P containing (w_{i_1}, f) (which exists as $(w_{i_1}, f) \in M$). In $M_{(w_{i_1}, f)}$ let (w, f_{j_2}) be the vertex containing w. As (w, f) is not a blocking pair in $M_{(w_{i_1}, f)}$ we must have $(w, f)(w, f_{j_2}) \in A_W$ and therefore $(w, f_{j_1})(w, f)(w, f_{j_2})$ is a path in $V(D(P))$.

Analogously, let $M_{(w, f_{j_1})}$ denote the stable matching in P containing (w, f_{j_1}). In $M_{(w, f_{j_1})}$ let (w_{i_2}, f) be the vertex containing f. As (w, f) is not a blocking pair in $M_{(w, f_{j_1})}$ we note that $(w_{i_1}, f)(w, f)(w_{i_2}, f)$ is a path in $V(D(P))$. Therefore Property (s) holds, a contradiction. □

Theorem 1. Let P be an instance of the matching problem and let P^* be its associated normal form. Then the following three statements are equivalent.

   (a) P has a unique stable matching.

   (b) The normal form of P, P^*, is acyclic.

   (c) In the normal form, P^*, every market participant's preference list a singleton.

Theorem 1'. Let P be an instance of the matching problem, and let $D(P)$ be the associated matching digraph, and let $D^*(P)$ be the matching digraph of the normal form. Then the following three statements are equivalent.

(a) P has a unique stable matching.

(b) $D^*(P)$ contains no cycle.

(c) $D^*(P)$ consists of n isolated vertices (that is, $|V(D^*(P))| = n$ and $|A(D^*(P))| = 0$).

The proof of Theorem $1'$ requires the following two additional lemmas, Lemma 3 and Lemma 4. Before stating and proving these Lemmas, we introduce the following notation. Let A_W^* denote the arcs from A_W with both endpoints still in $D^*(P)$. Analogously let A_F^* denote the arcs from A_F with both endpoints still in $D^*(P)$. That is $A(D^*(P)) = A_W^* \cup A_F^*$. Let $d_W^{*,+}(x)$ denote the number of arcs out of $x \in V(D^*(P))$ in A_W^* and analogously let $d_F^{*,+}(x)$ denote the number of arcs out of $x \in V(D^*(P))$ in A_F^*.

Lemma 3. Let P be an instance of the matching problem. Then the following two sets, M_W and M_F, are both stable matchings in P.

$$M_W = \{x \in V(D^*(P)) \mid d_W^{*,+}(x) = 0\}$$
$$M_F = \{x \in V(D^*(P)) \mid d_F^{*,+}(x) = 0\}$$

Proof. By Lemma 1, we note that there does exist some stable matching in $D^*(P)$. So for every $w \in W$ some vertex (w, f) belongs to $V(D^*(P))$. Let M_W be defined as above and note that for every $w \in W$ there exists exactly one vertex (w, f_w) in M_W. We first show that if $w_i, w_j \in W$ are distinct then f_{w_i} and f_{w_j} are distinct. For the sake of contradiction, assume that this is not the case, and $f_{w_i} = f_{w_j}$. Let $f = f_{w_i} = f_{w_j}$ and without loss of generality assume that $(w_i, f)(w_j, f) \in A(D^*(P))$. However this implies that we could perform Reduction $\mathcal{R}$ on (w_j, f), which would imply that we would have deleted (w_i, f), a contradiction. Therefore $f_{w_i} \neq f_{w_j}$ for all distinct w_i and w_j. Thus M_W is a matching.

By the proof of Lemma 2, every $(w, f) \in V(D^*(P))$ either belongs to M_W or is an in-neighbour of (w, f_w) which belongs M_W. Thus, there are no blocking pairs in M_W, which implies that M_W is indeed a stable matching in $D^*(P)$. By Lemma 1 M_W is therefore also a stable matching in P.

The fact that M_F is a stable matching in P can be shown analogously. $\square$

Lemma 4. Let P be an instance of the matching problem. If $A(D^*(P))$ is nonempty, then P contains at least two distinct stable matchings.

Proof. Let P be an instance that contains a stable matching, such that $A(D^*(P)) \neq \emptyset$. Without loss of generality assume that $(w, f_i)(w, f_j)$ is an arc in $A(D^*(P))$. Let $(w, f_k) \in V(D^*(P))$ be chosen such that $d_W^{*,+}((w, f_k)) = 0$ ($j = k$ is possible) and note that $i \neq k$. This implies that $(w, f_k) \in M_W$ (defined in Lemma 3).

If $M_W \neq M_F$ then P contains at least two distinct stable matchings (namely M_W and M_F). So, we may assume that $M_W = M_F$. Hence, $(w, f_k) \in M_F$. However this implies that (w, f_i) will be deleted by Reduction $\mathcal{R}$, as (w, f_i) is an in-neighbour of (w, f_k) and there is no arc $a \in A_F$ such that (w, f_k) is the tail of a. This is a contradiction to the fact that we cannot perform Reduction $\mathcal{R}$ on $D^*(P)$. □

Proof of Theorem 1′ (and hence of Theorem 1).

Proof. We first prove that (a) and (b) are equivalent. Assume that (b) holds. If (w_i, f_i) and (w_j, f_j) are distinct vertices in $V(D^*(P))$ then $w_i \neq w_j$ and $f_i \neq f_j$ as otherwise there would be an arc between the two vertices. So the n vertices in $V(D^*(P))$ form a matching of P. This matching is a stable matching, by Lemma 1′, as there are no blocking edges. Therefore (b) implies (a).

Now assume that (a) holds. By Lemma 4 we note that $A(D^*(P)) = \emptyset$. If $|V(D^*(P))| < n$, then P contains no stable matching (by Lemma 1′), a contradiction. If $|V(D^*(P))| > n$, then $A(D^*(P)) \neq \emptyset$, another contradiction (as there must then be two vertices in $D^*(P)$ with the same worker). So $|V(D^*(P))| = n$ and (b) holds.

Clearly (b) implies (c). We will show that (c) implies (b), so assume that (c) holds. If $A(D^*(P)) = \emptyset$, then (a) holds by Lemma 4, and therefore (b) also holds by the above. So we may assume that $A(D^*(P)) \neq \emptyset$. Let $p_0 p_1 \ldots p_r$ be a path in $D^*(P)$ with maximum number of vertices and, as $A(D^*(P)) \neq \emptyset$, we have $r \geq 1$. If $d_{D^*(P)}^+(p_r) > 0$, then the out-neighbour of p_r must lie in $V(P)$, due to the maximality of r, which gives us a cycle in $D^*(P)$, contradicting the fact that (c) holds. So $d_{D^*(P)}^+(p_r) = 0$. However, this implies that p_{r-1} will be deleted by Reduction $\mathcal{R}$, as $p_{r-1}p_r$ is an arc and there is no arc with tail p_r in either A_W or A_F. This is a contradiction to the fact that we cannot perform Reduction $\mathcal{R}$ on $D^*(P)$, which completes the proof.

□

B   Equivalent instances

We now sketch how an analyst can recover the full set of stable matching problems for which a given matching is the unique stable matching.

Begin with a particular matching, μ, and let us show how to "build up from scratch" the full collection of matching problems satisfying Definition 1 (i.e., $2n$ complete preference lists) with the property that the matching in question is the unique stable matching for each.[23]

The above is performed by effectively running IDUA "in reverse". Such a procedure works because any matching can be viewed as a collection of preference lists with only one participant per list. One can then augment the one-participant preference lists by continually adding participants in such a way that the IDUA procedure would have deleted the additional entries. While this does not yield a succinct condition on preferences like acyclicity (or any of the conditions provided by the papers listed in Footnote 3) and the procedure may be computationally intensive, it is exhaustive. That is, given any matching one can construct the entire set of equivalent instances, where equivalence is defined as possessing the same unique stable matching.

Lemma 1 ensures that if two instances of the matching problem possess the same normal form then they must possess the same set of stable matchings; but the reverse implication need not hold. However, parts (a) and (c) of Theorem 1 confirm that when an instance of the matching problem possesses a unique stable matching, the IDUA procedure collapses each participant's preference list to a singleton. And as such the reverse implication of Lemma 1 does hold for instances with a unique stable matching.

The fact above is useful as it allows the analyst to reconstruct the full set of complete preference lists for which a given matching is the unique stable matching. Let us elaborate on this. Consider Figure 3 and observe what happened in going from the second to last panel to the last panel. By the rules of the deletion procedure, the vertex $(3,2)$ was deleted and hence so were the arcs that made up the path $(2,2) \to (3,2), (3,2) \to (3,3)$ (which signify that $f_3 \succ_{w_3} f_2$ and $w_3 \succ_{w_2} w_2$). Note however, that the matching digraph in the final panel would have been the same if the path in the second to last panel had been the reverse, $(3,3) \to (3,2), (3,2) \to (2,2)$ ($f_2 \succ_{w_3} f_3$ and $w_2 \succ_{w_2} w_3$), or even if the path had instead been from vertex $(2,2)$ to $(3,3)$ via the vertex $(2,3)$ ($f_3 \succ_{w_2} f_2$ and $w_3 \succ_{w_3} w_2$).

This insight can be built upon. While $\mathcal{R}$ reduces a matching problem

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[23]We note that matching problems with incomplete preferences for which μ is the unique stable matching can also be recovered, but we omit the details.

without altering the set of stable matchings, one can define an operation, call it $\mathcal{R}^{-1}$, that expands a stable matching subproblem in such a way that the set of stable matchings is not altered. The inverse operation $\mathcal{R}^{-1}$ simply adds participants to preference lists in such a way that they would be deleted by an application of $\mathcal{R}$, though we note that $\mathcal{R}^{-1}$ is a correspondence and hence $\mathcal{R}^{-1}(D)$ denotes the set of all appropriate digraphs. Just as $\mathcal{R}$ can be applied repeatedly, the analyst can repeatedly apply $\mathcal{R}^{-1}$ until all $2n$ preference lists are complete, at which point the analyst has generated an instance of the matching problem for which the identified matching is the unique stable matching.

We will illustrate this inverse procedure for a specific example. When there are n workers and n firms, the number of possible instances where all participants have complete preferences is $(n!)^{2n}$, so for reasons of space we consider an example with $n = 2$. We let $W = \{w_1, w_2\}$ and $F = \{f_1, f_2\}$, and we consider the matching $\mu^* = \{(w_1, f_1), (w_2, f_2)\}$. We then ask: what is the full set of instances for which μ^* is the unique stable matching? Let us now show how to reverse engineer the set of preferences for which μ^* is the unique stable matching.

When $n = 2$, there are a total of $(2!)^{2 \times 2} = 16$ possible instances. Since the matching digraph for an instance with $n = 2$ is a 2×2 grid, and a minimum cycle has 4 vertices, there are only two possible instances that possess a cycle (one going "clockwise" and the other "anti-clockwise"). This means that for $n = 2$ there are 14 instances with a unique stable matching. Since there are only two possible matchings when $n = 2$, we can be sure that there are exactly 7 instances for which $\mu^* = \{(w_1, f_1), (w_2, f_2)\}$ is the unique stable matching. These are given in Figure 4 below, where each panel presents one such instance both in terms of preferences and using the digraph representation. In every digraph the vertices that comprise the original matching, μ^*, are shaded grey while the vertices that are added back are hollow. The arcs represent preferences.

The above illustrates how an analyst can "build up from scratch" the full set of equivalent instances of a the matching problem with a unique stable matching. We note that while the described procedure is computationally intensive, it is exhaustive. We conclude by pointing out that the procedure allows the identification of all instances for which the stated matching is the unique stable matching, and not merely those matching problems wherein every preference list is complete (which is what we have shown).

Figure 4: The 7 instances for which μ^* is the unique stable matching.

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